THE ECONOMIC THEORY OF INTEREST RATE AGGREGATION

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ABSTRACT

The empirical relationship between monetary aggregates, real output, and interest rates has featured prominently in studies of aggregate money demand, the monetary transmission mechanism, identification of monetary policy rules, and the business cycle. Such studies typically incorporate a single interest rate, but there is no consensus as to which is the relevant one. In this paper, we prove that if the conditions under which a monetary aggregate exists are imposed, then a derived interest rate aggregate can be defined. If the subutility function is homothetic, the interest rate aggregate can be tracked using index number methods. Including the interest rate aggregate in a model that contains a monetary aggregate is superior to choosing an individual interest rate in an ad hoc fashion because of internal consistency. We investigate the empirical relevance of our results by comparing the dual interest rate on money to individual interest rates using frequency domain techniques and by investigating money-income causality. We also generalize our results to allow for risk.

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THE EMPIRICAL RELATIONSHIP BETWEEN MONEY, REAL OUTPUT, AND INTEREST RATES has featured prominently in studies of aggregate money demand, the monetary transmission mechanism, identification of monetary policy rules, and the business cycle. No consensus exists as to which interest rate should be included in empirical models. Friedman and Kuttner (1993, pp. 193) state, "Although the inclusion of a short term interest rate in empirical work of this kind is now standard enough, there has been little discussion in the literature of just which short term rate is appropriate." Monetary assets are commonly aggregated to produce a single monetary quantity variable. Monetary assets can be both interest bearing or non-interest bearing, and interest bearing monetary assets earn different rates of return. Thus, it is not internally consistent to include only one interest rate or interest rate spread in an empirical model that contains a monetary aggregate.

Barnett (1978) proved that the user cost (price) of a monetary asset is the discounted present value of interest income foregone by holding the asset. Price aggregation theory can be used to define a user cost aggregate that is dual to the monetary quantity aggregate. The user cost aggregate can be defined as total expenditure on the portfolio divided by the quantity aggregate: a property called factor reversal. Using similar logic, Moore, Porter, and Small (1990) and Barnett and Xu (1998, 2000) define interest rate aggregates as the total interest on the portfolio divided by the value of the monetary quantity aggregate.[2] We prove that interest rate aggregation is generally inconsistent with microeconomic theory, unless it is derived from user cost aggregation. In this paper, we extend economic aggregation theory to derive an interest rate aggregate that is consistent with user cost aggregation. This paper provides microeconomic theory that justifies the factor reversal based definition in Barnett and Xu (1998, 2000).

The remainder of this paper is organized as follows: Section 1 reviews the theory of monetary asset and durable goods aggregation; Section 2 examines the consistency of user cost and interest rate aggregation, and derives the interest rate aggregate under perfect certainty; Section 3 provides empirical analysis of the interest rate aggregate; Section 4 extends the theoretical results to the case of risk; and Section 5 is a short conclusion.

1. PRICE AND QUANTITY AGGREGATION FOR DURABLE GOODS

We review aggregation theory in the context of a (representative) consumer's decision problem. Later results are also derived in this framework, but they can be extended to other contexts, such as production and financial intermediation.

1.1 User Costs for Durable Goods and Monetary Assets

Durable goods do not fully depreciate within a given decision period. The market price of a durable good is the discounted present value of all the service flows derived from the good. Diewert (1974, 1980) proved that the nominal user cost for durable good *i* is

(1) $$\pi_{it} = p_{it} - \frac{1-\delta_{it}}{1+R_t} p_{i,t+1},$$

where, p_{it} is the market price of the durable good in period *t*, δ_{it} is the depreciation rate of the good, and R_t is the risk free rate of return on an asset that can only be used to intertemporally transfer wealth. This asset is called the benchmark asset. The user cost is the opportunity cost of purchasing a unit of the good, using it for a single period, and then reselling the non-depreciated part of the unit at the prevailing price.

Barnett (1978, 1980, 1987) treated monetary assets as durable goods, and proved that the nominal user cost for monetary asset *i* is

(2) $$\pi_{it} = p_t^* \frac{R_t - r_{it}^M}{1+R_t},$$

where, r_{it}^M is the holding period yield on asset *i* in period *t*, and p_t^* is the value of a true cost of living index in period *t*. The real user cost, π_{it} / p_t^*, is the discounted present value of foregone interest $(R_t - r_{it}^M)$ paid on one dollar of deposits at the end of the holding period. The monetary asset user cost (2) is equivalent to (1) with depreciation rate

(3) $$\delta_{it} = \frac{p_{t+1}^* - p_t^*(1+r_{it}^M)}{p_{t+1}^*},$$

and $p_{it} = p_t^*$.[3] Similarly, (1)is equivalent to (2) if we define the holding period yield on durable good *i* as the capital gain on one dollar's worth of the good:

(4) $$r_{it}^D = \frac{p_{i,t+1}(1-\delta_{it}) - p_{it}}{p_{it}}.$$

Solving for δ_{it} and substituting into (1) produces

(5) $$\pi_{it} = \frac{p_{it}}{1+R_t}\left(R_t - r_{it}^D\right).$$[4]

We use the general notation π_{it} to denote either (2) or (5).

1.2 Price and Quantity Aggregation for Durable Goods and Monetary Assets[5]

The consumer is assumed to maximize a neoclassical utility function, *U*, subject to a budget constraint,

(6) $$\max_q \left\{ U(q) : \langle \pi, q \rangle = y \right\}$$

where *q* is a vector of *N* goods (possibly including non-durable goods and services, durable goods, and monetary assets), π is a vector of prices and user costs, *y* is total expenditure available for the *N* goods, and $\langle \pi, q \rangle = \sum_{i=1}^{N} \pi_i \cdot q_i$. The utility function *U* is *neoclassical* if it is (i) continuous, (ii) non-decreasing, and (iii) quasiconcave.

One interpretation of the problem given by (6) is that it represents a shadow problem within a more complex optimization problem. If *q* is a symmetrically weakly separable group in a more complex (possibly dynamic) utility maximization, then *y* is the total expenditure on the *N* goods implied by the optimal solution to the more complex problem, and *U* is the subutility function for the weakly separable group. Under mild regularity conditions on the utility function for a more complex problem in which *U* is nested, the optimal solution from (6) will coincide with the optimal solution for the *N* goods from the more complex problem; this property is called *strong decentralizability*.[6] Strong decentralizability is the essential concept underlying the economic approach to aggregation. It implies that the consumer can be viewed as optimally allocating expenditure within a sector using only intrasector prices and the amount of total expenditure allocated to the sector. The solution to the sectoral problem produces an aggregate measure of the utility derived from the sector. If, in addition, *q* is homothetically weakly separable within a more complex problem, then multi-stage allocation theory can be used to simplify the optimization.[7]

The economic approach to aggregation uses microeconomic decision theory to define quantity and price aggregates. The quantity aggregate is the distance function that is dual to *U*, $d(q, \tilde{u})$, defined implicitly by $U(q / d(q, \tilde{u})) = \tilde{u}$. The exact Malmquist (1953) quantity index $M(q_1, q_0, \tilde{u}) = d(q_1, \tilde{u}) / d(q_0, \tilde{u})$, is the quantity aggregate normalized on a base period, where q_0 and q_1 are the quantity vectors in periods 0 and 1. The user cost aggregate is the expenditure function, $e(\pi, \tilde{u})$, that is dual to *U*, defined by $e(\pi, \tilde{u}) = \min_q \left\{ \langle \pi, q \rangle : U(q) = \tilde{u} \right\}$. The exact Konüs

(1939) user cost index $K(\pi_1,\pi_0,\tilde{u}) = e(\pi_1,\tilde{u})/e(\pi_0,\tilde{u})$ is the user cost aggregate normalized on a base period, where π_0 and π_1 are the price vectors in periods 0 and 1.

Linear homogeneity of the user cost (quantity) aggregate is a desirable property because it implies that a proportional change in all components of π (q) will result in the same proportional change in the aggregate. The regularity conditions on U imply that the aggregates $e(\pi,\tilde{u})$ and $d(q,\tilde{u})$ are linearly homogeneous in π and q respectively.

In general, $e(\pi,\tilde{u})\cdot d(q,\tilde{u}) \le \langle \pi, q\rangle$ for any π, q, and $\tilde{u}$.[8] If the optimal solution to the expenditure minimization given π and $\tilde{u}$ is proportional to q, then π and q are said to be *direct conjugates* for $\tilde{u}$.[9] Direct conjugacy implies that $e(\pi,\tilde{u})\cdot d(q,\tilde{u}) = \langle \pi, q\rangle$; this property is called *strong factor reversal*. Diewert (1981, page 177) proved that if U is neoclassical and the consumer is expenditure minimizing a referent utility level, $\hat{u}$, will exist such that $M(q_1,q_0,\hat{u})\cdot K(\pi_1,\pi_0,\hat{u}) = \langle \pi_1, q_1\rangle / \langle \pi_0, q_0\rangle$; this property is called *weak factor reversal*. Weak factor reversal requires optimization.

In general, the Malmquist and Konüs indexes depend on $\tilde{u}$; they are invariant to $\tilde{u}$ if and only if U is homothetic. If U is linearly homogeneous, $M(q_1,q_0,\tilde{u}) = \dfrac{d(q_1,\tilde{u})}{d(q_0,\tilde{u})} = \dfrac{U(q_1)}{U(q_0)}$ and $K(\pi_1,\pi_0,\tilde{u}) = \dfrac{e(\pi_1,1)}{e(\pi_0,1)}$.[10] Under linear homogeneity, the aggregates strongly factor reverse and the exact indexes weakly factor reverse for any referent utility level.

These results provide the basic properties of user cost and quantity aggregates: (i) The user cost (quantity) aggregate is a function only of user costs (quantities); (ii) The user cost (quantity) aggregate is positive linearly homogeneous in user costs (quantities); and (iii) Weak or strong factor reversal.

Barnett (1980, 1987) used these results to define quantity and user cost aggregates for a group of monetary assets. Barnett, Fisher, and Serletis (1992) and Anderson, Jones, and Nesmith (1997) review the relevant literature on monetary aggregation theory.

2. Interest Rate Aggregation

Barnett (1987, pp. 161-163) has argued that it is "a hazardous venture" to define aggregates of variables other than prices (user costs) and quantities. In order to define an economic aggregate, we must assume that the consumer is optimizing some objective function subject to a constraint. If the solution to an alternative optimization differs from the solution to (6), then the agent can only maximize one of the two objective functions. We elucidate Barnett's critique for interest rate aggregation in Section 2.1, and provide a solution in Section 2.2.

2.1 Consistency of Interest Rate Aggregation

User cost aggregation theory is based on the utility maximization in (6). If U is linearly homogeneous, then $Q = U(q^*)$ and $\Pi = e(\pi,1) = \langle \pi, q^* \rangle / U(q^*)$ define dual quantity and user cost aggregates, where $q^* = \phi(\pi, y)$, and ϕ is the vector of Marshallian demand functions.

An interest rate aggregate can be defined, using standard aggregation techniques, if the agent solves a decision problem of the following form:

(7) $$\max_q \left\{ F(q) : \langle r, q \rangle = X \right\},$$

where F satisfies the same regularity conditions as U, r is a vector of interest rates, and X is total interest income required on the portfolio. If F is linearly homogeneous, then $Q^F = F(q^{**})$ and $r^F = r \cdot q^{**} / F(q^{**})$ define dual quantity and interest rate aggregates, where $q^{**} = \psi(r, X)$ is the optimal solution to (7). This decision is inconsistent with the utility maximization in (6) unless the optimal quantity vectors, q^* and q^{**}, are equal for all feasible values of the constraint variables. A necessary condition for consistency of the two decisions is that the optimal quantity vector from (6) must satisfy the constraint in (7), so that $\langle r, q^* \rangle = X$. Theorem 1 proves that (6) and (7) are inconsistent, for at least some values of the constraint variables.

THEOREM 1*: Let U and F be positively linearly homogeneous, differentiable, neoclassical functions. If there exist* π_0, r_0, y_0, *and* p_0^* *in the positive orthant such that* $\phi(\pi_0, y_0) = \psi\left(r_0, \langle r_0, \phi(\pi_0, y_0) \rangle\right)$, *then there exist* π_1, r_1, y_1, *and* p_1^* *in the positive orthant such that* $\phi(\pi_1, y_1) \neq \psi\left(r_1, \langle r_1, \phi(\pi_1, y_1) \rangle\right)$, *where, for* i = 1,…,*N*, $\pi_{0i} = p_0^*(R_0 - r_{0i})/(1+R_0)$ *and* $\pi_{1i} = p_1^*(R_1 - r_{1i})/(1+R_1)$.

PROOF: Without loss of generality, we can assume that $\exists i, j$ such that $r_{0i} \neq r_{0j}$.[11] Assume that for π_o, r_0, y_0, and p_0^*, the optimal solutions to (6) and (7) coincide, such that $\phi(\pi_0, y_0) = \psi(r_0, r_0 \cdot \phi(\pi_0, y_0))$. The first order conditions for (6) imply that

$$\frac{U_i(\phi(\pi_0, y_0))}{U_j(\phi(\pi_0, y_0))} = \frac{\pi_{0i}}{\pi_{0j}} = \frac{R_0 - r_{0i}}{R_0 - r_{0j}}.$$

Choose any $R_1 \neq R_0$. This implies that $R_1 - r_{0i} / R_1 - r_{0j} \neq R_0 - r_{0i} / R_0 - r_{0j}$, because $r_{0i} \neq r_{0j}$. Define $\pi_1 = (\pi_{11}, ... \pi_{1N})$ through the relations $\pi_{1i} = p_0^*(R_1 - r_{0i})/(1+R_1)$ for i = 1,…,N. The first order conditions for (6) imply that

$$\frac{U_i(\phi(\pi_1, y_1))}{U_j(\phi(\pi_1, y_1))} = \frac{R_1 - r_{0i}}{R_1 - r_{0j}} \neq \frac{R_0 - r_{0i}}{R_0 - r_{0j}}$$

for any y_1. This implies that $\phi(\pi_0, y_0)$ is not proportional to $\phi(\pi_1, y_1)$ for any y_0 and y_1, because U is linearly homogeneous. Now define $y_1 = y_0 \cdot \langle r_0, \phi(\pi_0, 1)\rangle / \langle r_0, \phi(\pi_1, 1)\rangle$. We show that π_1, $r_1 = r_0$, y_1, and $p_1^* = p_0^*$ will suffice to prove the proposition. The following derivation shows that $\phi(\pi_1, y_1)$ and $\phi(\pi_0, y_0)$ earn the same interest income, $\langle r_0, \phi(\pi_1, y_1)\rangle = \langle r_0, \phi(\pi_1, 1)\rangle \cdot y_1 = \langle r_0, \phi(\pi_1, 1)\rangle \cdot y_0 \cdot \frac{\langle r_0, \phi(\pi_0, 1)\rangle}{\langle r_0, \phi(\pi_1, 1)\rangle} = \langle r_0, \phi(\pi_0, 1)\rangle \cdot y_0 = \langle r_0, \phi(\pi_0, y_0)\rangle$.

Thus, $\psi\left(r_0, \langle r_0, \phi(\pi_1, y_1)\rangle\right) = \psi\left(r_0, \langle r_0, \phi(\pi_0, y_0)\rangle\right) = \phi(\pi_0, y_0) \neq \phi(\pi_1, y_1)$. *Q.E.D.*

The intuition for this result is that different user cost vectors can be constructed from the same vector of holding period yields, because the benchmark rate is not uniquely determined by the vector of holding period yields. Under linear homogeneity, these two user cost vectors result in optimal quantity vectors that are not proportional, unless all holding period yields are equal. Consequently, given a pair of solutions to (6) and (7) that are consistent, an inconsistent pair of solutions can be constructed. If the benchmark rate is uniquely determined from the interest rate vector then the construction followed in the proof cannot be carried out, and a function F could be defined such that (6) and (7) will be consistent. If the holding period yields of the N assets are always equal, it can be shown that F may be defined as U.

As previously discussed, Diewert's proof of weak factor reversal assumes that the agent is optimizing in both the current and base periods. An interest rate aggregate can be defined using weak factor reversal only if (7) is valid. Consequently, we have proven that weak factor reversal can not be used to define an interest rate aggregate, although it may be a property of an aggregate defined in an alternative manner.

2.2 Interest Rate Aggregation

Barnett and Xu (1998, 2000) define the rate of return that is dual to the monetary quantity aggregate as the total holding period yield of the monetary portfolio divided by the quantity aggregate; a definition based on strong factor reversal.

We provide a direct definition of the interest rate aggregate and prove that it satisfies three properties that are analogous to the properties of the user cost aggregate, and a fourth property that connects the interest rate aggregate to the user cost aggregate. Our definition also allows for durable goods, so our definition is not restricted to monetary assets .[12] The user cost aggregate is the minimum expenditure necessary to achieve a referent level of utility:

(8) $$e(\pi_t, \tilde{u}) = \langle \pi_t, h(\pi_t, \tilde{u}) \rangle .$$

where $h(\pi_t, \tilde{u})$ denotes a vector of Hicksian demand functions. The user cost formula can be substituted into (8) to obtain the following expression for the user cost aggregate:

(9) $$e(\pi_t, \tilde{u}) = \sum p_t^* \frac{R_t - r_{it}}{1 + R_t} h_i(\pi_t, \tilde{u}) .$$

We define $r_t^* = \langle r_t, h(\pi_t, \tilde{u}) \rangle$ and $R_t^* = R_t \cdot \sum_{i=1}^{N} h_i(\pi_t, \tilde{u})$. Substitution into (9) gives us the following expression:

(10) $$e(\pi_t, \tilde{u}) = p_t^* (\frac{R_t^* - r_t^*}{1 + R_t}) .$$

Equation (10) proves that the user cost aggregate has a mathematical form analogous to the individual user cost; next, we argue that the economic meaning is also analogous.

We define the interest rate aggregate as r_t^*, which is the holding period yield on the expenditure minimizing portfolio. The user cost aggregate is the discounted present value of the difference between the holding period yield on the expenditure minimizing portfolio, r_t^*, and the maximum holding period yield that could have been earned on a portfolio of equal initial value, R_t^*.

The interest rate aggregate also satisfies properties that are weaker than, but analogous to, those of the user cost aggregate.

THEOREM 2: *Assume that U satisfies the regularity conditions, and, for convenience, that the solutions to the expenditure minimization are functions rather than correspondences. Then: (i)* r_t^* *is a function only of holding period yields,* $(R_t, r_{1t}, ..., r_{Nt})$*, including the benchmark rate; (ii)* r_t^* *is positively linearly homogeneous in interest rates (or rates of return), in the sense that* $\lambda r_t^*(R_t, r_{1t}, ..., r_{Nt}) = r_t^*(\lambda R_t, \lambda r_{1t}, ..., \lambda r_{Nt})$ *for any* $\lambda > 0$*, and any referent utility level* $\tilde{u}$*; (iii) if* π_t *and* q_t *are direct conjugates for* $\tilde{u}$ *then* $r_t^* = \langle r_t, q_t \rangle / d(q_t, \tilde{u})$*; and (iv)*

$$e(\pi_t, \tilde{u}) = p_t^* \left(\frac{R_t^* - r_t^*}{1 + R_t}\right).$$

PROOF:

(i) Let, $k = (1 + R_t)/p_t^*$, so that $k\pi_t = (R_t - r_{1t}, ..., R_t - r_{Nt})$. Positive homogeneity of degree zero of the Hicksian demand functions implies that $r_t \cdot h(\pi_t, \tilde{u}) = r_t \cdot h(k\pi_t, \tilde{u})$ for any $k > 0$, and consequently, r_t^* is a function only of $(R_t, r_{1t}, ..., r_{Nt})$.

(ii) is demonstrated by the following: $r_t^*(\lambda R_t, \lambda r_{1t}, ..., \lambda r_{Nt}) = \langle \lambda r_t, h(\lambda \pi_t, \tilde{u}) \rangle = \lambda \cdot \langle r_t, h(\pi_t, \tilde{u}) \rangle = \lambda r_t^*(R_t, r_{1t}, ..., r_{Nt}) \;\; \forall \lambda > 0$.

(iii) If U is a positively linearly homogeneous aggregator function, then $d(q_t, \tilde{u}) = U(q_t)/\tilde{u}$ and $h(\pi_t, \tilde{u}) = h(\pi_t, 1)\tilde{u}$. In this case, $d(q_t, \tilde{u}) r_t^* = d(q_t, \tilde{u}) \cdot \langle r_t, h(\pi_t, \tilde{u}) \rangle = (U(q_t)/\tilde{u}) \cdot \langle r_t \cdot h(\pi_t, 1) \rangle \cdot \tilde{u}$ $= U(q_t) \cdot \langle r_t, h(\pi_t, 1) \rangle = \langle r_t, h(\pi_t, U(q_t)) \rangle$. Duality theory implies that $\langle r_t, h(\pi_t, U(q_t)) \rangle = \langle r_t, q_t \rangle$, if $e(\pi_t, U(q_t)) = \langle \pi_t, q_t \rangle$. If U is not positively linearly homogeneous, but π_t and q_t are direct conjugates for $\tilde{u}$, then $h(\pi_t, \tilde{u}) = q_t / d(q_t, \tilde{u})$.[13] In this case,

$$r_t^* = \langle r_t, h(\pi_t, \tilde{u}) \rangle = \langle r_t, q_t \rangle / d(q_t, \tilde{u}).$$

(iv) Shown by equation (10). *Q.E.D.*

An main implication of the argument in Section 2.1 is that the interest rate aggregate must be a function of the benchmark rate. If the benchmark rate changes, holding all other rates constant, the interest rate aggregate will also change, and properties (i) and (ii) are, therefore, weaker than the corresponding properties for the user cost aggregate. Property (iii) is strong factor reversal at points of direct conjugacy. Under linear homogeneity of the aggregator function, the interest rate aggregate will factor reverse on any level set, assuming optimization. Theorem 1 proves that the interest rate aggregate is not produced from an optimization in the form of (7), however, so the revealed preference arguments used in Diewert's (1981, page 177) proof cannot be used to show weak factor reversal of the interest rate aggregate in the absence of homotheticity. Property (iv) proves that the user cost aggregate is equal to the difference between the benchmark rate and the interest rate aggregate discounted to present value, where the benchmark rate is rescaled.[14]

Property (iv) is the main economic justification for defining r_t^* as the interest rate aggregate. If multi-stage budgeting is valid, the agent's full decision problem can be rewritten in terms of the quantity and user cost aggregates. In such a case, the quantity aggregate will be indistinguishable to the agent from an elementary good, with price given by the user cost aggregate. Property (iv) proves that the user cost aggregate can be expressed as the spread between the benchmark rate of return and the interest rate aggregate, so that the full decision problem could also be rewritten in terms of the quantity and interest rate aggregates, provided that the rescaled benchmark rate is used where appropriate. Therefore, if multi-stage budgeting is feasible, the quantity aggregate is indistinguishable to the agent from an elementary durable good, which earns a rate of return given by the interest rate aggregate.

3. Empirical Results

In this section, we investigate the empirical relevance of our results by comparing the dual interest rate for a monetary aggregate (which we think is the clearest application of our theory) to individual interest rates and by investigating money-income causality. We are assuming that the vector of monetary assets is weakly separable, or, more generally, form a decentralizable sector. We chose to use the standard M2 grouping of monetary assets to facilitate comparison to earlier studies, rather than test for a weakly separable or decentralizable sector.[15] We can provide results at other levels of aggregation upon request. The results reported in this section are not very sensitive to the level of aggregation.

3.1 Monetary Interest Rate Aggregate versus Individual Interest Rates

Under linear homogeneity, the interest rate aggregate can be approximated by dividing the total interest on the portfolio by an index number approximation to the quantity aggregate. Anderson, Jones, and Nesmith (1997) constructed monthly estimates of the aggregation theoretic monetary quantity and price aggregates for the period 1960-1997. We extended these estimates to the period 1950-1997.[16] We constructed estimates of the aggregation theoretic interest rate aggregates using this data for the M2 level of aggregation.

The aggregation theoretic dual interest rate is compared with four individual interest rates: three month Treasury bill rates, six month commercial paper rates, the overnight federal funds rate, and the implicit rate of return on small denomination time deposits.[17] We introduce the following notation: RM2 is the interest rate aggregate for the M2 level of aggregation, RTB is the 3-month (secondary market) Treasury bill rate, RCP is the 6-month commercial paper rate, RFF is the overnight Federal Funds rate, and RSTD is the dual rate on small denomination time deposits. The addition of a "D" prior to the name of the variable indicates the first difference of the variable. All variables are measured monthly, but some rates are available for slightly different sample periods.[18]

The estimated correlation function between the interest rate aggregate, DRM2, and the individual interest rates DRTB, DRCP, DRFF, and DRSTD are provided in Table I. The table provides contemporaneous correlations in part A, and the estimates for ±6 lags in part B. The results indicate that there is substantial contemporaneous correlation between the interest rate aggregate and all of the other interest rates. The interest rate aggregate is most correlated with the interest rate on small denomination deposits, which seems reasonable, because small time deposits are a component of M2. Of the remaining interest rates, the correlation is slightly higher between DRM2 and DRTB than DRCP and DRFF. In all cases, the contemporaneous correlation exceeds the correlation at all leads and lags.

We also performed coherence analysis: a frequency domain version of correlation analysis. The squared coherence between two time series is the percentage of total power in one series that can be explained by a linear regression on the other series at a given frequency. We estimate coherence using non-parametric methods over a set of frequencies. Squared coherence is the frequency domain analogue of correlation. Koopmans (1975) and Carter (1993) are standard references on the subject. A detailed discussion of our estimators is available from the authors upon request.

The logarithms of the interest rates all have strong spectral peaks near the zero frequency. The first difference filter, which has gain function $|G(f)|^2 = 4\sin^2(\pi f)$, is used to eliminate these peaks. Preliminary analysis indicated that these first differenced rates do not have white power spectra, but that an AR(12) pre-whitening filter was sufficient to flatten the spectra for each of the rates. The qualitative results of the coherence analysis are not sensitive to pre-whitening, once the main spectral peak is removed.[19] We estimate coherence for the pre-whitened series, using overlapped blocks of length 18. In all tables, frequencies are reported in units of cycles/month.[20] The tables contain estimated coherence $\hat{\rho}_{XY}(f)$, upper and lower bounds for 95 percent confidence intervals, and F-tests for zero coherence. Squared coherence, $\hat{\rho}^2_{XY}(f)$, is the proportion of power at a given frequency in one series that can be explained by a linear regression on the other at frequency *f*.

There is a general pattern in the coherence between the Treasury bill, commercial paper, and overnight federal funds rates and the dual interest rate aggregate. In most cases, the coherence exhibits a propensity to diminish at higher frequencies, and is often statistically indistinguishable from zero at high frequencies. The coherence between the dual interest rate aggregate and DRTB is reported in Table II.[21] The squared coherence between DRM2 and DRTB indicates that 41 percent of the power in the two series at the lowest frequency can be explained by a linear regression of one on the other. Squared coherence is lower at higher frequencies and is statistically indistinguishable from zero at the two highest frequencies at the 5% level. The coherence between the dual interest rate aggregate and DRCP is reported in Tables III. The squared coherence between DRM2 and DRCP at the lowest frequency indicates that 30 percent of the power in each series can be explained by a linear regression of one series on the other. Coherence is lower at all higher frequencies and is not significantly different from zero at the two highest frequencies. The coherence between the dual interest rate aggregate and DRFF is reported in Tables IV. The pattern of declining coherence is again evident, although the value of the estimated coherence is not significantly different from zero at several additional frequencies.

The Treasury bill rate is often interpreted as the alternative risk free rate of return for transaction media, such as currency.[22] The analysis shows that the Treasury bill and commercial paper rates should be regarded as short term interest rates that are highly correlated with the rate of return on a broad monetary aggregate at low frequencies rather than as alternative risk free rates of return for medium of exchange.

It seems reasonable that the rate of return on an asset in M2 would be more coherent with the rate of return on money than the other rates, because the other rates do not correspond to assets in M2. The results are reported in Tables V. The estimated coherence between DRM2 and DRSTD is much higher than that of the other rates. Squared coherence exceeds .8 for all frequencies, and does not diminish at the higher frequencies. This result confirms our hypothesis.

3.2 Granger Causality

Eichenbaum and Singleton (1986) and Sims (1980) have argued that the fraction of the variance of output explained by innovations in money is reduced by the inclusion of interest rates in the model. Stock and Watson (1989) emphasized the sensitivity of these results to the techniques used to detrend the series, and concluded that money does have significant marginal predictive value in models that contain interest rates. Friedman and Kuttner (1992, 1993) dispute these results because they are sensitive to both sample period and choice of short term interest rate. Specifically, Friedman and Kuttner (1993) argue that either including data through 1990 or using the commercial paper rate instead of the Treasury bill rate is sufficient to render money insignificant as a predictor of future output.[23]

We reinvestigate this issue using aggregation theoretic measures of money and interest rates. We introduce the following notation, in addition to the notation introduced in Section 3.1. MSIM2 is the logarithm of a superlative index number approximation to the monetary quantity aggregate at the M2 level of aggregation (in per-capita terms), PM2 is the logarithm of the dual user cost aggregate, M2 is the Federal Reserve's M2 aggregate, and Y is the logarithm of per-capita private net national product. All variables are measured quarterly. Some short comment is necessary here. According to theory, the monetary quantity aggregate is the sub-utility function for a weakly separable grouping of assets evaluated at the optimum. This is not observable, so we must use an index number to approximate it. The variable MSIM2 is a superlative index number approximation constructed by the Federal Reserve Bank of St. Louis, see Anderson, Jones, and Nesmith (1997) for details. The Federal Reserve's monetary aggregates are more commonly used in empirical research, but they are not superlative index numbers. We refer the interested reader to Anderson, Jones, and Nesmith (1997) and the references therein.

We replicate Granger causality tests for the official M2 monetary aggregate, as defined by the Federal Reserve Board for purposes of comparison. We tested the hypothesis that DM2 Granger causes DY in a bivariate system and in multivariate systems that contained either DRTB or DRCP over four time periods 1954:1-1997:4, 1954:1-1979:3, 1970:3-1997:4, and 1984:1-1997:4.[24] The first three sample periods are extensions of the ones studies by Friedman and Kuttner (1992). The results are reported in Table VI. In the bivariate systems, DM2 is significant in all samples except 1984:1-1997:4, but interest rates are also marginally insignificant during this period. The multivariate results are also not robust to the sample period. The monetary aggregate is significant and the interest rates are insignificant in the sample 1954:1-1979:3. The opposite conclusion is obtained in the sample 1970:3-1997:4. None of the variables are marginally significant over the sample 1984:1-1997:4. The results over the full sample depend on which interest rate is used. The commercial paper rate dominates the monetary aggregate over the full sample, but the Treasury bill rate does not.[25] These results are similar to Friedman and Kuttner's results.

Bivariate and multivariate tests of the hypothesis that DMSIM2 Granger causes DY are reported in Table VII. The bivariate results for DMSIM2 are almost identical to the ones for DM2. The monetary aggregates Granger cause DY for all sample periods except 1984:1-1997:4. The multivariate Granger causality tests are run using both the dual interest rate and dual price aggregates. The monetary aggregates dominate the price/interest rate variable over all sample periods, and are marginally significant during every period, except during the post 1983 sample.

The use of the aggregation theoretic variables clears up some of the issues presented above. The dual price and interest rate aggregates defined by monetary aggregation theory are a more conceptually sound measure of aggregate "financial market price information". Aggregation theoretic monetary aggregates can be used to predict DY over any sample period except 1984-1997. The monetary aggregates Granger cause DY even if aggregation theoretic price/interest rate aggregates are included in the model.

4. Stochastic Generalizations

Barnett (1995), Barnett and Liu (1995), Barnett, Liu, and Jensen (1997), and Jones and Nesmith (2000) generalized monetary aggregation theory to include risk bearing monetary assets.[26]

In this section, we derive risk-adjusted user costs for durable goods, prove a decentralizability theorem, and define the generalized risk-adjusted interest rate aggregate. These results show that price and interest rate aggregation theory can be generalized under risk, and that the user cost aggregate has the same interpretation as the user cost of an elementary durable good.

4.1 Risk-Adjusted User Costs for Durable Goods

We proceed from a generalization of the infinite horizon household model of Barnett and Liu (1995) and Barnett, Liu, and Jensen (1997). Define *Y* to be the household's survival set, which is a compact subset of the (*m*+*n*)+2 real non-negative orthant. Let the consumption possibility set in period s, $S(s)$, be defined as follows:

$$S(s) = \{(q_s, c_s, A_s) \in Y : \quad p_s^* c_s = \sum_{i=1}^{m+n} [(1+r_{i,s-1}) p_{i,s-1} q_{i,s-1} - p_{is} q_{i,s}] + (1+R_{s-1}) p_{s-1}^* A_{s-1} - p_s^* A_s \}$$

where, c_s is aggregate non-durable goods and services, p_s^* is the price dual to c_s, A_s is the stock of the benchmark asset, $q_s = (q_{1s}, ..., q_{(m+n),s})$ is an $(m+n)$-vector of durable goods, p_{is} is the market price of durable good *i*, and r_{is} is the holding period on durable good *i*.[27] The vector of durable goods includes both *n* monetary assets and *m* durable consumer goods, where $r_{is} = r_{is}^M$ and $p_{is} = p_s^*$ for monetary assets, and $r_{is} = r_{is}^D$ for durable consumer goods. Prices and interest rates are stochastic processes. $r_{i,t-1}^M$ is known at the beginning of period *t*, and r_{it}^M is realized at the end of period *t*. p_{it} and p_t^* are known at the beginning of period *t*, $p_{i,t+1}$ and p_{t+1}^* realized at the beginning of period *t*+1 (or equivalently at the end of period *t*). This implies that $r_{i,t-1}^D$ is known at the beginning of period *t*, and r_{it}^D is realized at the beginning of period *t*+1.

The households decision, at time *t*, is to choose the deterministic point (q_t, c_t, A_t) and the stochastic processes (q_s, c_s, A_s), s = *t*+1 to 4

$$\max \left\{ u(q_t, c_t) + E_t [\sum_{s=t+1}^{\infty} (\tfrac{1}{1+\xi})^{s-t} u(q_s, c_s)] \right\}, \tag{11}$$

subject to $(q_s, c_s, A_s) \in S(s)$, for all $s = t$ to ∞ and the transversality condition $\lim_{s\to\infty} E_t (\tfrac{1}{1+\xi})^{s-t} A_s = 0$.[28] The transversality condition rules out perpetual borrowing at the benchmark rate. We assume that the (*m*+*n*) durable quantities are homothetically weakly separable, such that $u(q_t, c_t) = V(U(q_t), c_t)$, where *U* is linearly homogeneous.

We denote the optimal period s controls by $u_s^* = (q_s^*, A_s^*)$. We denote the period s states by $x_s = (q_{s-1}, A_{s-1}, \varphi_s)$, where φ_s denotes the vector of prices and interest rates that were realized at the beginning of period *s*. Assuming that an interior solution exists, the Euler equations imply that

(12)
$$V_U(u_t^*, x_t)U_i(q_t^*) = \frac{1}{1+\xi} E_t\left[\frac{p_{it}(R_t - r_{it})}{p_{t+1}^*} V_c(u_{t+1}^*, x_{t+1})\right]$$

(13)
$$V_c(u_t^*, x_t) = \frac{1}{1+\xi} E_t\left[\frac{p_t^*(1+R_t)}{p_{t+1}^*} V_c(u_{t+1}^*, x_{t+1})\right],$$

where $V_U = \dfrac{\partial V}{\partial U}$, $U_i = \dfrac{\partial U}{\partial q_i}$, and $V_c = \dfrac{\partial V}{\partial c}$.[29]

We define the contemporaneous risk-adjusted nominal user cost of durable good *i* as

(14)
$$\pi_{it} = p_t^* \frac{V_U(u_t^*, x_t)U_i(q_t^*)}{V_c(u_t^*, x_t)} = p_{it} \frac{E_t[((R_t - r_{it})/p_{t+1}^*)V_c(u_{t+1}^*, x_{t+1})]}{E_t[((1+R_t)/p_{t+1}^*)V_c(u_{t+1}^*, x_{t+1})]}.$$ [30]

We note that the risk-adjusted user costs are deterministic. If the agent is risk neutral, the utility function, *V*, is linear and V_c is a constant. Therefore, the risk-neutral nominal user cost is given by

(15)
$$\pi_{it}^{RN} = p_{it} \frac{E_t[((R_t - r_{it})/p_{t+1}^*)]}{E_t[((1+R_t)/p_{t+1}^*)]} = p_{it} \frac{E_t[\hat{R}_t] - E_t[\hat{r}_{it}]}{1 + E_t[\hat{R}_t]},$$

where $\hat{R}_t = \dfrac{p_t^*(1+R_t) - p_{t+1}^*}{p_{t+1}^*}$ and $\hat{r}_{it} = \dfrac{p_t^*(1+r_{it}) - p_{t+1}^*}{p_{t+1}^*}$.

Theorem 1 in Barnett and Liu (1995) proves that the risk-adjusted monetary user cost is the sum of the risk neutral user costs and a risk premium. Theorem 3 extends this result to the general durable goods case. The proof is omitted because it closely follows Barnett and Liu (1995, Theorem 1, pp. 10-11).

THEOREM 3: *The risk-adjusted real user cost has the form:*

(16)
$$\pi_{it} = \pi_{it}^{RN} + (A_{it} - B_{it}),$$

where $A_{it} = \dfrac{p_{it}(1 - \Pi_{it}^{RN}/p_{it})}{(1+\xi)} \dfrac{\mathrm{Cov}(\hat{R}_t, V_c(u_{t+1}^*, x_{t+1}))}{V_c(u_t^*, x_t)}$ *and* $B_{it} = \dfrac{p_{it}}{(1+\xi)} \dfrac{\mathrm{Cov}(\hat{r}_{it}, V_c(u_{t+1}^*, x_{t+1}))}{V_c(u_t^*, x_t)}$.[31]

Under risk neutrality, marginal utility of consumption is constant, the covariances in (16) are identically zero, and the risk-adjusted real user cost equal the risk neutral user cost. Under perfect certainty, interest rates, prices, and the marginal utility of consumption are deterministic and $\pi_{it} = p_{it}(R_t - r_{it})(1+R_t)$, which is the perfect certainty user cost. The risk-adjusted user costs are, therefore, strict generalizations of both the perfect certainty and risk neutral user costs. Theorem 3 proves that the risk-adjusted real user costs are equal to the sum of the risk neutral user costs and $(A_{it} - B_{it})$. The latter component can be interpreted as a risk-premia.[32] If $\hat{r}_{it}$ is deterministic then A_{it} is the risk premium for durable good *i*. If the interest rate on the durable good is positively correlated with the marginal utility of consumption then it can lower the risk in the consumption stream, and the risk premium will be less than A_{it}.

4.2 User Cost and Interest Rate Aggregation Under Risk

U is a well-defined aggregator function, under risk. Positive linear homogeneity of *U* implies that

$$\sum_{i=1}^{m+n} \pi_{it} q_{it}^* = \sum_{i=1}^{m+n} p_t^* \frac{V_U(u_t^*, x_t) U_i(q_t^*)}{V_c(u_t^*, x_t)} q_{it}^* = p_t^* \frac{V_U(u_t^*, x_t)}{V_c(u_t^*, x_t)} \sum_{i=1}^{m+n} U_i(q_t^*) q_{it}^* = p_t^* \frac{V_U(u_t^*, x_t)}{V_c(u_t^*, x_t)} U(q_t^*).$$[33]

The ratio of partial derivatives, $p_t^* V_U(u_t^*, x_t) / V_c(u_t^*, x_t)$, factor reverses with the quantity aggregate to the optimal nominal expenditure on durable goods and monetary assets. Theorem 4 proves that the risk-adjusted user costs are shadow prices, which decentralize the agent's decision. Theorem 4 also proves that unit expenditure function is the dual user cost aggregate under risk.

THEOREM 4 (DECENTRALIZABILITY UNDER RISK): *Assume that U is a linearly homogeneous, differentiable, aggregator function. There exist shadow prices* $(\zeta_1, ..., \zeta_{m+n})$ *such that the optimal solution to the following shadow decision,*

$$\max_{q_t} \{U(q_t) : \sum_{i=1}^{m+n} \zeta_i q_{it} = \sum_{i=1}^{m+n} \zeta_i q_{it}^* \in Y'\}, \tag{17}$$

is q_t^*, *where* Y' *is the (compact) projection of Y onto an (m+n) dimensional subspace having components* q_t. *The shadow prices are the risk-adjusted user costs, and the dual real user cost aggregate is* $e(\pi_{1t}, .., \pi_{(m+n),t}, 1) = p_t^* \dfrac{V_U(u_t^*, x_t)}{V_c(u_t^*, x_t)}$.

PROOF: If the vector of shadow prices, $(\zeta_1,...,\zeta_{m+n})$, rationalizes q_t^*, then the first order conditions imply that $U_i(q_t^*)/U_j(q_t^*) = \zeta_i/\zeta_j$. Equations (12) and (13) imply that these first order conditions will be satisfied if $\zeta_i = \pi_{it}$ for all i = 1,...,(m+n). The second order conditions are satisfied because U is an aggregator function, and q_t^* satisfies the budget constraint. Duality implies that $e(\pi_{1t},..,\pi_{(m+n),t},U(q_t^*)) = \sum_{i=1}^{m+n}\pi_{it}q_{it}^*$.

Also, $e(\pi_{1t},..,\pi_{(m+n),t},U(q_t^*)) = e(\pi_{1t},..,\pi_{(m+n),t},1)U(q_t^*)$, by linear homogeneity of U, and thus $e(\pi_{1t},..,\pi_{(m+n),t},1) = \sum_{i=1}^{m+n}\pi_{it}q_{it}^*/U(q_t^*) = p_t^*\dfrac{V_U(u_t^*,x_t)}{V_c(u_t^*,x_t)}$. *Q.E.D.*

The shadow decision (17) decentralizes the agent's decision, and is used to prove that the unit expenditure function, $e(\pi_{1t},..,\pi_{(m+n),t},1)$, is the generalized user cost aggregate under risk. The homothetic weak separability assumptions maintained in this model would be sufficient to treat this shadow decision as a second stage decision, under perfect certainty. In general, this is not true under risk, because the risk-adjusted user costs are functions of $u_t^* = (q_t^*, A_t^*)$. In other words, in order to decentralize the decision it is necessary to know the optimal durable goods quantity vector. Consequently, the decentralized decision cannot be a second stage decision. Under risk neutrality, however, the risk-adjusted user costs depend only the expectations of prices and interest rates, and Barnett (1995) proved that two stage budgeting theory is applicable under risk neutrality. Therefore, Theorem 4 proves that the agent's decision is decentralizable under risk aversion, but that additive price aggregation is not valid except under risk neutrality.

Theorems 3 and 4 proved that the risk-adjusted user costs are of the form: $\pi_{it} = p_{it}(E_t[\hat{R}_t] - E_t[\hat{r}_{it}])\big/(1+E_t[\hat{R}_t]) + (A_{it} - B_{it})$, and that the dual user cost aggregate is $e(\pi_{1t},..,\pi_{(m+n),t},1)$. Analogous to the derivation in (8)-(10), the risk-adjusted nominal user cost aggregate is given by the following expression:

$$\text{(18)} \qquad e(\pi_{1t},..,\pi_{(m+n),t},1) = \frac{(E_t[\hat{R}_t^*] - E_t[\hat{r}_t^*])}{1+E_t[\hat{R}_t]} + (A_t - B_t)$$

where

$$\hat{R}_t^* = \hat{R}_t \sum_{i=1}^{n+m} p_{it} h_i\ (\pi_{1t},...,\pi_{(n+m),t},1)$$

$$\hat{r}_t^* = \sum_{i=1}^{n+m} p_{it} \hat{r}_{it}\, h_i\ (\pi_{1t},...,\pi_{(n+m),t},1)$$

$$A_t \ = \sum_{i=n+1}^{n+m} A_{it} h_i\ (\pi_{1t},...,\pi_{(n+m),t},1)$$

$$= \left(1 - \left(\frac{E_t\,[\hat{R}_t^*] - E_t\,[\hat{r}_t^*]}{1 + E_t\,[\hat{R}_t\,]} \Big/ \sum_{i=n+1}^{n+m} p_{it} h_i\ (\pi_{1t},...,\pi_{(n+m),t},1)\right)\right) \frac{\text{Cov}(\hat{R}_t^*, V_c\ (u_{t+1}^*, x_{t+1}))}{(1+\xi)V_c\ (u_t^*, x_t\,)}$$

$$B_t \ = \sum_{i=n+1}^{n+m} B_{it} h_i\ (\pi_{1t},...,\pi_{(n+m),t},1) \ = \ \frac{\text{Cov}(\hat{r}_t^*, V_c\ (u_{t+1}^*, x_{t+1}))}{(1+\xi)V_c\ (u_t^*, x_t\,)}.$$ [34]

The expression in (18) is equivalent to the perfect certainty expression (10) adjusted for risk. Under risk neutrality, $(A_t - B_t)$ is identically zero. Thus, $(E_t\,[\hat{R}_t^*] - E_t\,[\hat{r}_t^*])\big/(1 + E_t\,[\hat{R}_t\,])$ is the risk neutral nominal user cost aggregate, and the risk-adjusted nominal user cost aggregate is equal to the sum of the risk neutral user cost aggregate and an aggregate risk premium, $(A_t - B_t)$, which is the total risk premium on the expenditure minimizing portfolio. The expected holding period yield on the expenditure minimizing portfolio, $E_t\,[\hat{r}_t^*]$, is a generalization of the perfect certainty interest rate aggregate to the case of risk aversion.[35]

5. CONCLUSION

It is self evident that numerous interest rates exist in modern economies. In order to speak of ‘the interest rate’ in a manner analogous to discussing ‘the inflation level’, those rates must be aggregated. We prove that if the conditions under which a monetary aggregate exists are imposed (weak separability or more generally decentralizability), then a derived interest rate aggregate exists. Prior research on interest rate aggregates relied on strong factor reversal as a definition, rather than a property. Such methods represent the test or axiomatic approach to constructing index numbers rather than the economic approach. See Diewert (1981, 1987). We believe our definition is more satisfying, both because it is more general and because it is grounded in the economic approach. Our definition does, however, satisfy strong factor reversal under homotheticity and is in that sense similar to a price aggregate.

Developing the theory of interest rate aggregation immediately suggests numerous empirical applications. This is standard fare in aggregation theory. As Deaton (1981) states, “The theory and measurement of economic index numbers present side by side some of the most difficult and abstruse theory with the most immediately practical issues of everyday measurement." The inclusion of an index that tracks the interest rate aggregate, in models that contain monetary aggregates, is internally consistent, whereas choosing to include an individual interest rate in an ad hoc fashion is not. We investigated the empirical relevance of our results by comparing the dual interest rate on money to individual interest rates and by investigating money-income causality, using frequency domain techniques. This investigation shed some light on debates in the literature. In particular, we find that commonly used individual interest rates have high correlation with an M2 level interest rate aggregate at low frequencies. We also found that there is little evidence to support the conventional wisdom that the predictive power of monetary aggregates declines when interest rates are included in an econometric model.

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ENDNOTES

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[2] Moore, Porter, and Small (1990) use the official sum monetary aggregate in their definition. Barnett and Xu (1998, 2000) use the exact aggregation-theoretic monetary aggregate.

[3] See Fisher, Hudson, and Pradhan (1993).

[4] The quantity of durable good i is converted from dollars to units by the current market price of the good, whereas nominal monetary asset stocks are converted to real monetary asset stocks by the current value of the true cost of living index, p_t^*.

[5] In this section, we draw heavily from Diewert (1981), Blackorby, Primont, and Russell (1978), and Barnett (1987).

[6] Theorem 5.2 in Blackorby, Primont, and Russell (1978, page 188) states the precise regularity conditions.

[7] This property is called *additive price aggregation*, and Theorem 5.6 in Blackorby, Primont, and Russell (1978, page 206) states the precise conditions under which it is applicable.

[8] Blackorby, Primont, and Russell 1978, page 28

[9] Formally, π and q are direct conjugates for $\tilde{u}$, if the $\tilde{u}$ level set is supported at $q / d(q, \tilde{u})$ by a hyperplane with normal π, where $e(\pi, \tilde{u}) = 1$, see Blackorby, Primont and Russell (1978, page 28).

[10] Linearly homogeneity of U implies that $d(q, \tilde{u}) = U(q) / \tilde{u}$ and $e(\pi, \tilde{u}) = e(\pi, 1)\tilde{u}$.

[11] If the only consistent points are ones with all own rates being equal, then simply choose any user cost vector generated from unequal own rates, and the proposition will be proven.

[12] In fact, non-durable goods and services could also be included, but doing so causes some problems in that homothetic changes in rates of return are ruled out by including assets that always have a gross rate of return of zero.

[13] See Blackorby, Primont, and Russell (1978, page 28). We are assuming that the solution to expenditure minimization is unique. If it is not, the equality will hold for some element of the Hicksian demand correspondence.

[14] The re-scaling converts the benchmark rate into the same units as the interest rate aggregate. The interest rate aggregate is a weighted sum of the component interest rates, but the weights do not sum to one. The benchmark rate must, therefore, be multiplied by the sum of the weights, $\Sigma_{i=1}^{N} h_i(\pi_t, \tilde{u})$, in order to be directly comparable in levels to the interest rate aggregate.

[15] Blackorby, Primont and Russell (1977, 1978 pp. 290-316) note the theoretical difficulties in testing for weak separability with flexible functional forms. A Monte Carlo comparison by Barnett and Choi (1989) of tests based on flexible functional forms and also nonparametric tests found that none performed adequately. Note that weak

separability is a stronger concept than decentralizability; see Theorem 5.3 in Blackorby Primont, and Russell (1978, pp. 189).

[16] Details of this extension are available on request.

[17] The Treasury bill, commercial paper, and federal funds rates are available from the Federal Reserve Board. The "implicit rate of return on small time deposits" is an implicit rate that is dual to the user cost of small time deposits. The user cost is a unilateral index number constructed from a variety of available small time deposit rates. All rates are converted to one-month holding period yields on a bond interest basis. See Anderson, Jones, and Nesmith (1997)

[18] For example, the three month Treasury bill rate is available for 1951:2-1997:12, whereas the six month commercial paper rate is available from 1960:2-1997:8.

[19] The use of pre-whitening filters to eliminate bias in the estimation is discussed in Brillinger and Rosenblatt (1967). It is not necessary to difference the data if pre-whitening is used, because the AR filter can difference the data. In practice, we found that an AR(1) filter on the levels essentially differenced all of the interest rates. The coherence analysis of the pre-whitened levels is consistent with the results reported here. A unit root analysis of the interest rates also indicated that the levels should be first differenced. We do not report these results, however, because differencing can be justified purely on the grounds that they pre-whiten the series.

[20] In all tables, the estimates for the zero frequency are omitted. The differencing filter has a gain of zero at this frequency.

[21] The results are similar if the 6-month Treasury bill rate is used instead of the 3- month Treasury bill rate.

[22] Moore, Porter, and Small (1990) for example use the Treasury bill rate as the alternative rate of return for M2.

[23] Stock and Watson perform their tests over the period 1960-1985 using the 3-month Treasury bill. Friedman and Kuttner (1992) performed a similar analysis over three different sub-samples. They find that the evidence for marginal significance of money as a predictor of the change in output is strongest for the sample 60:2-79:3, is significantly weaker for the sample 60:2-90:4, and disappears altogether for the period 70:3-90:4.

[24] The samples are different for the systems that contain the commercial paper rate. The commercial paper rate is available for 60:2-97:3. In the systems that contain DRCP, the sample is truncated at either endpoint as appropriate.

[25] The latter conclusion is also true using the sample 60:2-97:3, which is used in the systems containing DRCP.

[26] Barnett (1995) proved that the category sub-utility function for a weakly separable group of monetary assets is a quantity aggregator function in a well-defined sense, under risk. Barnett and Liu (1995) and Barnett, Liu, and Jensen (1997) derive risk-adjusted user costs, which equal the Barnett (1978) user costs, under perfect certainty. Jones and Nesmith (2000b) provided conditions under which solutions to the expected decision problems exist.

[27] For expository purposes, we assume that non-durable goods and services are weakly separable. Aggregate consumption is the category sub-utility function of these goods and services, and the dual price is the unit expenditure function.

[28] We assume that the vector of interest rates and prices is a first-order vector Markov process and E_t denotes integration against the transition probabilities of that process. If only monetary assets were under consideration the decision would be exactly the one considered in Barnett and Liu (1995), and Barnett, Liu, and Jensen (1997).

[29] See Jones and Nesmith (2000) for conditions which guarantee that a unique, but possibly non-interior, solution exists.

[30] This definition agrees with the Barnett and Liu (1995) definition for monetary assets, and has similar properties as will be demonstrated.

[31] $\mathrm{Cov}(x, y) = E_t[(x - E_t[x])(y - E_t[y])] = E_t[xy] - E_t[x]E_t[y]$

[32] See Barnett, Liu, and Jensen (1997) and Barnett and Liu (1995) who interpret the risk-premium based on the standard consumption capital asset pricing model.

[33] The last equality is by Engel's law. See Barnett and Liu (1995, equation 4.18).

[34] As in the perfect certainty derivation, *h* denotes the vector of Hicksian demand functions, which solve the expenditure minimization dual to (17). The definition of the risk-adjusted user costs is substituted into the expenditure function to obtain (18).

[35] For symmetry, the perfect certainty interest rate aggregate could be defined in terms of $\hat{r}_t$.

TABLE I

PART A – CONTEMPORANEOUS CORRELATIONS

	DRM2	DRTB	DRCP	DRFF	DRSTD
DRM2	1.	.62	.52	.53	.96
DRTB		1.	.90	.75	.55
DRCP			1.	.80	.42
DRFF				1.	.46
DRSTD					1.

PART B – CROSS CORRELATIONS BETWEEN DRM2 AND INDIVIDUAL RATES (± 6 LAGS)

	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
DRTB	-.16	.06	-.06	-.16	-.04	.11	.62	.43	-.09	.01	-.03	-.01	-.02
DRCP	-.15	.09	-.06	-.13	.03	.07	.52	.39	-.13	.01	-.03	-.02	.04
DRFF	-.12	.01	-.13	-.06	.02	.22	.53	.29	-.05	.02	-.03	-.07	-.07
DRSTD	-.01	.04	-.17	-.05	-.09	.14	.96	.15	-.06	-.03	-.16	.03	-.03

The Cross correlation is given by the formula $\rho_{xy}(k) = \frac{\sum (x_t - \bar{x})(y_{t-k} - \bar{y})}{\sqrt{\sum (x_t - \bar{x})^2 \sum (y_{t-k} - \bar{y})^2}}$, where x is DRM2 and y is DRTB, DRCP, DRFF, or DRSTD and k runs from –6 to 6.

TABLE II

COHERENCE BETWEEN DRM2 AND DRTB3

1951:02 - 1997:12

Freq.	95% LB	ρ_{XY}	95% UB	ρ_{XY}^2	FSTAT	P{FSTAT<F(2,EDF-2)}	
0.0556	0.4909	0.6392	0.7416	0.4086	30.6018	(0.0000)	**
0.1111	0.4612	0.6160	0.7238	0.3794	27.0760	(0.0000)	**
0.1667	0.3183	0.5001	0.6330	0.2501	14.7663	(0.0000)	**
0.2222	0.4089	0.5743	0.6916	0.3298	21.7951	(0.0000)	**
0.2778	0.2428	0.4360	0.5812	0.1901	10.3965	(0.0001)	**
0.3333	0.2995	0.4843	0.6204	0.2346	13.5704	(0.0000)	**
0.3889	0.3032	0.4874	0.6229	0.2376	13.8008	(0.0000)	**
0.4444	-0.0371	0.1805	0.3623	0.0326	1.4911	(0.2308)	
0.5000	-0.1506	0.0677	0.2588	0.0046	0.2039	(0.8159)	

FSTAT ~ F(2,EDF-2) under the null that coherence is zero.

** significant at the 5% level, * significant at 10% level

TABLE III

COHERENCE BETWEEN DRM2 AND DRCP6

1960:02 - 1997:08

Freq.	95% LB	ρ_{XY}	95% UB	ρ_{XY}^2	FSTAT	P{FSTAT<F(2,EDF-2)}
0.0556	0.3448	0.5417	0.6780	0.2934	14.7079	(0.0000) **
0.1111	0.2359	0.4522	0.6084	0.2045	9.1053	(0.0003) **
0.1667	0.1080	0.3412	0.5185	0.1164	4.6660	(0.0125) **
0.2222	0.2721	0.4824	0.6322	0.2328	10.7438	(0.0001) **
0.2778	0.1260	0.3572	0.5317	0.1276	5.1808	(0.0080) **
0.3333	0.2403	0.4559	0.6113	0.2078	9.2912	(0.0003) **
0.3889	0.2616	0.4737	0.6253	0.2244	10.2477	(0.0001) **
0.4444	-0.1142	0.1316	0.3374	0.0173	0.6241	(0.5387)
0.5000	-0.0453	0.1990	0.3974	0.0396	1.4602	(0.2392)

FSTAT ~ F(2,EDF-2) under the null that coherence is zero.

** significant at the 5% level, * significant at 10% level

TABLE IV

COHERENCE BETWEEN DRM2 AND DRFF

1955:08 - 1997:12

Freq.	95% LB	ρ_{XY}	95% UB	ρ_{XY}^2	FSTAT	P{FSTAT<F(2,EDF-2)}	
0.0556	0.4101	0.5835	0.7038	0.3405	20.5772	(0.0000)	**
0.1111	0.2958	0.4907	0.6316	0.2408	12.6406	(0.0000)	**
0.1667	-0.0101	0.2185	0.4046	0.0477	1.9979	(0.1424)	
0.2222	0.1766	0.3890	0.5495	0.1513	7.1056	(0.0015)	**
0.2778	-0.0671	0.1635	0.3558	0.0267	1.0945	(0.3397)	
0.3333	0.1011	0.3217	0.4935	0.1035	4.6010	(0.0129)	**
0.3889	-0.0070	0.2214	0.4071	0.0490	2.0539	(0.1350)	
0.4444	0.0605	0.2846	0.4619	0.0810	3.5124	(0.0346)	**
0.5000	-0.0403	0.1895	0.3790	0.0359	1.4843	(0.2329)	

FSTAT ~ F(2,EDF-2) under the null that coherence is zero.

** significant at the 5% level, * significant at 10% level

TABLE V

COHERENCE BETWEEN DRM2 AND DRSTD

1960:02 - 1997:12

Freq.	95% LB	ρ_{XY}	95% UB	ρ^2_{XY}	FSTAT	P{FSTAT<F(2,EDF-2)}
0.0556	0.8463	0.9033	0.9365	0.8160	157.0933	(0.0000) **
0.1111	0.9057	0.9414	0.9617	0.8862	275.6824	(0.0000) **
0.1667	0.9140	0.9467	0.9652	0.8961	305.5948	(0.0000) **
0.2222	0.9096	0.9438	0.9634	0.8908	288.9195	(0.0000) **
0.2778	0.9006	0.9382	0.9596	0.8802	260.1571	(0.0000) **
0.3333	0.8997	0.9376	0.9593	0.8791	257.5348	(0.0000) **
0.3889	0.9514	0.9700	0.9806	0.9410	564.6390	(0.0000) **
0.4444	0.9165	0.9482	0.9663	0.8991	315.5489	(0.0000) **
0.5000	0.9482	0.9681	0.9793	0.9371	527.9172	(0.0000) **

FSTAT ~ F(2,EDF-2) under the null that coherence is zero.
** significant at the 5% level, * significant at 10% level

TABLE VI:

GRANGER CAUSALITY TESTS FOR FEDERAL RESERVE BOARD M2 MONETARY AGGREGATE

(various time periods)

Excluded Variable	1954:01-1997:04	1954:01-1979:03	1970:03-1997:04	1984:01-1997:04
System variables: DY, DM2				
DM2	7.7514 (.0000)	5.5026 (.0005)	5.6693 (.0004)	.8423 (.5061)
System variables: DY, DM2, DRTB				
DM2	3.6803 (.0067)	3.3161 (.0139)	1.8445 (.1265)	.4570 (.7668)
DRTB	1.9067 (.1118)	1.5431 (.1965)	2.8541 (.0277)	.2286 (.9209)
System variables: DY, DM2, DRCP				
DM2	1.7946 (.1334)	3.3059 (.0158)	.8291 (.5099)	.4158 (.7962)
DRCP	2.7479 (.0308)	.3235 (.8612)	3.4467 (.0112)	.7824 (.5430)

TABLE VII

GRANGER CAUSALITY TESTS FOR AGGREGATION-THEORETIC MONETARY AGGREGATE

(various time periods)

Excluded Variable	1954:01-97:04	1954:01-1979:03	1970:03-1997:04	1984:01-1997:04
System variables: DY, DMSIM2				
DMSIM2	6.5900 (.0001)	5.1234 (.0009)	4.7774 (.0015)	.5812 (.6779)
System variables: DY, DMSIM2, DRM2				
DMSIM2	6.2268 (.0001)	3.9658 (.0052)	4.6591 (.0018)	.4206 (.7929)
DRM2	.6354 (.6379)	.3857 (.8183)	1.7196 (.1518)	.1741 (.9505)
System variables: DY, DMSIM2, DPM2				
DMSIM2	3.3113 (.0122)	2.2956 (.0652)	2.8574 (.0275)	.7321 (.5751)
DPM2	1.8238 (.1267)	.4211 (.7930)	1.9408 (.1098)	.4266 (.7885)