COHERENCE ANALYSIS

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We provide a short discussion of coherence analysis. See Carter (1993) for more details.

Definitions

Let $X(t)$ and $Y(t)$ be zero mean weakly stationary stochastic processes. The *power spectrum* of X is defined as the Fourier transform of the second order moment sequence $c_{XX}(\tau) = E[X(t)X(t+\tau)]$:

$$P_{XX}(f) = \sum_{\tau=-\infty}^{\infty} c_{XX}(\tau) e^{-i2\pi f\tau}\ ,\ |f| < 1/2\ .^{\text{i}}$$

Similarly, the *cross power spectrum* is defined as the Fourier transform of the second order cross moment sequence $c_{XY}(\tau) = E[X(t)Y(t+\tau)]$:

$$P_{XY}(f) = \sum_{\tau=-\infty}^{\infty} c_{XY}(\tau) e^{-i2\pi f\tau}\ ,\ |f| < 1/2\ .$$

Absolute summability of the second order moment functions implies that the auto and cross power spectra exist and are well defined.

The auto and cross spectra can be interpreted using the spectral representations of *X* and *Y*:

$$X(t) = \int_{-1/2}^{1/2} e^{i2\pi ft} dZ_X(f)$$

$$Y(t) = \int_{-1/2}^{1/2} e^{i2\pi ft} dZ_Y(f)\ ,$$

where $Z_X(f)$ and $Z_Y(f)$ are orthogonal increment processes with the following properties:

$$E[dZ_X(f)\overline{dZ_X(f)}] = P_{XX}(f)df\ ,\ E[dZ_X(f)\overline{dZ_X(g)}] = 0$$

$$E[dZ_X(f)\overline{dZ_Y(f)}] = P_{XY}(f)df\ \text{, and}\ E[dZ_X(f)\overline{dZ_Y(g)}] = 0\ .^{\text{ii}}$$

The power spectrum describes the contribution to the expectation of the product of two Fourier components whose frequencies are the same. The integral of the power spectrum is equal to the variance of the sequence, $c_{XX}(0)$, consequently, the power spectrum can be interpreted as a decomposition of the variance by frequency. The cross spectrum is used to determine the degree of linear association between the two stochastic processes at different frequencies.

The *coherence*, $\rho_{XY}(f)$, between X and Y is the normalized modulus of the cross power spectrum:

$$\rho_{XY}(f) = \frac{|P_{XY}(f)|}{P_{XX}^{1/2}(f)P_{YY}^{1/2}(f)}, \; |f| < 1/2 .$$

Intuitively, squared coherence is a measure of correlation between the two time series at each frequency, and as such is a frequency domain analogue of the coefficient of correlation. The following argument justifies this interpretation.

The output of a time invariant linear filter of X(t), $L(X(t)) = \sum_{i=-\infty}^{\infty} h_i \, X(t-i)$, will have spectral representation $L(X(t)) = \int_{-1/2}^{1/2} e^{i2\pi ft} H(f) dZ_X(f)$, where $H(f)$ is the Fourier transform of the filter coefficients $\{h_i\}$. The variance of the residual between $Y(t)$ and the output of the filter is given by $\sigma^2 = E[|Y(t) - L(X(t))|^2]$. Substituting the spectral representations into the definition yields the following expression:

$$\begin{aligned}\sigma^2 &= E[|\int_{-1/2}^{1/2} e^{i2\pi ft} dZ_Y(f) - \int_{-1/2}^{1/2} e^{i2\pi ft} H(f) dZ_X(f)|^2] \\ &= \int_{-1/2}^{1/2} P_{YY}(f)df + \int_{-1/2}^{1/2} P_{XX}(f)|H(f)|^2 - H(f)P_{XY}(f) - \overline{H(f)P_{XY}(f)}df\end{aligned}$$

which is minimized by choosing the transfer function $\tilde{H}(f) = \frac{P_{YX}(f)}{P_{XX}(f)}$. The minimized variance of the residual is given by the expression $\tilde{\sigma}^2 = \int_{-1/2}^{1/2} P_{YY}(f)(1-\rho_{XY}^2(f))df$. Thus, squared coherence, $\rho_{XY}^2(f)$, is the portion of power, at frequency *f*, of either stochastic process that can be explained by its linear regression on the other. Coherence is invariant to linear filtration and is symmetric in the sense that $\rho_{XY}(f) = \rho_{YX}(f)$.

The *phase angle* between *X* and *Y*, $\theta_{XY}(f)$, is given by the expression:

$$P_{XY}(f) = |P_{XY}(f)| e^{i\theta_{XY}(f)} .$$

The alternative expression

$$\frac{P_{YX}(f)}{P_{XX}(f)} = \frac{|P_{YX}(f)|}{P_{XX}(f)} e^{i(-\theta_{XY}(f))}$$

shows that the phase angle is the negative of the phase shift of the linear filter that provides the best approximation of *Y* in terms of *X*. The ratio of the phase angle to the frequency, $\tau_{XY}(f) = \theta_{XY}(f)/f$ is a measure of the time lead or lag between the *X* and *Y* frequency by frequency.

X and *Y* could be coherent because they are both coherent with one or more stochastic processes. This problem is dealt with by filtering both *X* and *Y* on a set of additional processes to eliminate as much power as possible in both series and estimating the coherence of the residuals. Let $Z_1(t), \dots, Z_m(t)$ be *m* zero mean weakly stationary stochastic processes. Let $\tilde{X}_{Z_1,\dots,Z_m}(t)$ denote the output of the linear filter that minimizes the mean square error for *X*, and similarly let $\tilde{Y}_{Z_1,\dots,Z_m}(t)$ denote the output of the linear filter that minimizes mean square error for *Y*. The *partial coherence* between *X* and *Y*, given the *Z* processes, $\rho_{XY|Z_1,\dots,Z_m}(f)$, is the coherence between the residual processes $X^U(t) = X(t) - \tilde{X}_{Z_1,\dots,Z_m}(t)$ and $Y^U(t) = Y(t) - \tilde{Y}_{Z_1,\dots,Z_m}(t)$. It is important to note that partial coherence can exceed coherence or vice verse.

Spectral Estimation

Let $\{X_0, \dots, X_{N-1}\}$ and $\{Y_0, \dots, Y_{N-1}\}$ be finite data records of the two time series. Segment these records into *K* (possibly) overla

$$\{X_0^k, \dots, X_{L-1}^k\} = \{X_{(k-1)D}, \dots, X_{(k-1)D+L-1}\}$$

$$\{Y_0^k, \dots, Y_{L-1}^k\} = \{Y_{(k-1)D}, \dots, Y_{(k-1)D+L-1}\}, \text{ k } = 1, \dots, \text{K}.$$

The total number of frames $K = \dfrac{(N-L)}{D} + 1$. If $D = L$ then the frames are non-overlapped, and if $D = L/2$ then the frames overlap by 50 percent.[iii]

The finite Fourier transforms of the *k*th standardized frame are defined as

$$d_{X^k}(f_n) = \sum_{s=0}^{L-1} X_s^k e^{-i2\pi f_n [s+(k-1)L]}$$

$$d_{Y^k}(f_n) = \sum_{s=0}^{L-1} Y_s^k e^{-i2\pi f_n [s+(k-1)L]}$$

where $f_n = n/L$, for n = 0,…,*L*/2. The second order *periodograms* for the *k*th frame are

$$I_k^{XX}(f_n) = (1/L)\,|\,d_{X^k}(f_n)\,|^2$$

$$I_k^{YY}(f_n) = (1/L)\,|\,d_{Y^k}(f_n)\,|^2$$

$$I_k^{XY}(f_n) = (1/L) d_{X^k}(f_n) \overline{d_{Y^k}(f_n)}\,.$$

The periodograms for the *K* frames are averaged to obtain consistent estimators of the auto and cross spectra. This estimation method is called the faded overlapping segment (FOS) method described in Welch (1967, 1977), Groves and Hannan (1968), Kay and Marple (1981), and Carter and Nuttall (1980a,b). The power spectrum estimators are defined as

$$\hat{P}_{XX}(f_n) = \frac{1}{K}\sum_{k=1}^{K} I_k^{XX}(f_n)$$

$$\hat{P}_{YY}(f_n) = \frac{1}{K}\sum_{k=1}^{K} I_k^{YY}(f_n)\,.$$

The cross-spectrum estimator is defined as

$$\hat{P}_{XY}(f_n) = \frac{1}{K}\sum_{k=1}^{K} I_k^{XY}(f_n)\,.$$

Each frame is adjusted is two ways: a trapezoidal data taper is applied to each frame to lower bias due to side lobe distortion, and the frame elements are standardized using the mean and standard deviation of the appropriate elements from all *K* frames.[iv]

Welch (1967), and Groves and Hannan (1968) derived a general form for the equivalent degrees of freedom, ν, for the FOS estimator. The expression (from Welch) is as follows:

$$\nu = 2\frac{E^2[P(f_n)]}{Var[\hat{P}(f_n)]} = \frac{2K}{1+2\sum_{j=1}^{K-1}\rho(j) - 2\sum_{j=1}^{K-1}(j/K)\rho(j)} \quad ,$$

where $\rho(j) = [\sum_{k=0}^{L-1} a(k)a(k+jD)]^2 \Big/ [\sum_{k=0}^{L-1} a(k)^2]^2$ and *a*(*k*) *k* = 1,…,*L*-1 are the taper coefficients. If the frames do not overlap (*D* = *L*) then $\nu = 2N/L$, whereas if the frames overlap by 50 percent (*D* = *L*/2) then $\nu \approx 3.27N/L$. Therefore, overlapping leads to substantial reduction in the variance of the estimators. See Welch (1967), Groves and Hannan (1968), Carter and Nuttall (1980a), and Carter, Knapp, and Nuttall (1973) for detailed discussions. These studies recommend 50 percent overlap. Yuen (1977) has proposed an alternative method of improving stability. The alternative procedure doubles the length of each segment by adding zeros, but without tapering the segments. The spectral estimates are frame averages computed without overlap, which are then smoothed using a Hanning window.[v]

Asymptotic Distributions and Confidence Intervals for Spectral Estimates

Let $\hat{P}_{XX}(f)$, $\hat{P}_{YY}(f)$, and $\hat{P}_{XY}(f)$ be consistent estimators of the auto and cross spectra with equivalent degrees of freedom, ν.

Coherence

The coherence, $\rho_{XY}(f)$, between X and Y is estimated by

$$\hat{\rho}_{XY}(f) = \frac{|\hat{P}_{XY}(f)|}{\hat{P}_{XX}^{1/2}(f)\hat{P}_{YY}^{1/2}(f)}.$$

The confidence interval for the estimated coherence is derived from the standardized asymptotic distribution of $\phi(f) = \tanh^{-1}(\hat{\rho}_{XY}(f))$. In particular, $\phi(f)$ is distributed asymptotically normal. Standardizing by mean and variance of $\phi(f)$ yields the following expression:

$$\frac{\tanh^{-1}(\hat{\rho}_{XY}(f)) - \tanh^{-1}(\rho_{XY}(f)) - (\nu-2)^{-1}}{(\nu-2)^{-1}} \sim N(0,1).$$

This asymptotic distribution can be used for sample sizes that exceed 20, provided $.4 < \rho^2(f) < .95$.[vi] See Koopmans (1975, pp. 282-283) and Enochson and Goodman (1965).

The upper and lower bounds for a $(1-\alpha)$ percent confidence interval are given by

$$\overline{\rho} = \tanh\{\tanh^{-1}(\hat{\rho}) + u_{\alpha/2}(\nu-2)^{-1/2} - (\nu-2)^{-1}\}$$

$$\underline{\rho} = \tanh\{\tanh^{-1}(\hat{\rho}) - u_{\alpha/2}(\nu-2)^{-1/2} - (\nu-2)^{-1}\},$$

where $u_{\alpha/2}$ is the $\alpha/2$ cutoff for the standard normal distribution. If the estimated coherence is low, an F-test for zero coherence can be carried out. Under the hypothesis that coherence is zero the statistic $\frac{(\nu-2)\hat{\rho}_{XY}^2(f)}{2(1-\hat{\rho}_{XY}^2(f))}$ is distributed F(2, $\nu-2$), see Koopmans (1975, pp. 284).

Phase Angle

The phase angle, $\theta_{XY}(f)$, between X and Y is estimated by

$$\hat{\theta}_{XY}(f) = \tan^{-1}\left(\frac{\operatorname{Im}\hat{P}_{XY}(f)}{\operatorname{Re}\hat{P}_{XY}(f)}\right).$$

The confidence interval for the phase angle is derived from a joint confidence interval for the gain and phase of the linear filter between *X* and *Y* that minimizes residual variance, see Goodman (1957) and Koopmans (1975, pp. 287). A $(1-\alpha)$ percent confidence interval is given by

$$\hat{\theta}_{XY}(f) \pm \sin^{-1}\left(\left(\frac{2F_{2,\nu-2}(\alpha)}{\nu-2}\right)^{1/2} \frac{(1-\hat{\rho}_{XY}^2(f))^{1/2}}{\hat{\rho}_{XY}(f)} \frac{\hat{P}_{XX}(f)}{\hat{P}_{YY}(f)}\right)$$

where $F_{2,\nu-2}(\alpha)$ is the upper α cutoff point of the F distribution. The length of the confidence interval is inversely related to estimated coherence, and goes to zero as coherence approaches one. If estimated coherence is so low that the term inside the inverse sine exceeds one, then the confidence interval is $\hat{\theta}_{XY}(f) \pm \pi/2$.

Partial Coherence

The estimated partial coherence, $\hat{\rho}_{XY|Z_1,\ldots,Z_m}(f)$, between X and Y is given by

$$\hat{\rho}_{XY|Z_1,\ldots,Z_m}(f) = \frac{|\hat{P}^U_{XY|Z_1,\ldots,Z_m}(f)|}{(\hat{P}^U_{XX|Z_1,\ldots,Z_m}(f)\hat{P}^U_{YY|Z_1,\ldots,Z_m}(f))^{1/2}},$$

where $\hat{P}^U_{j,k|Z_1,\ldots,Z_m}(f) = \hat{P}_{j,k}(f) - \hat{\mathbf{P}}_{j,Z}(f)\hat{\mathbf{P}}_{Z,Z}(f)^{-1}\overline{\hat{\mathbf{P}}_{k,Z}(f)}$ for *j*,*k* = *X*,*Y*. The 1 by *m* matrix, $\hat{\mathbf{P}}_{j,Z}(f)$, has elements $\hat{P}_{j,Z_k}(f)$ (*k* = 1,…,*m*), and the *m* by *m* matrix, $\hat{\mathbf{P}}_{Z,Z}(f)$, has elements $\hat{P}_{Z_j,Z_k}(f)$ (*j*,*k* = 1,…,*m*).

The confidence intervals and F-tests for zero partial coherence are the same as for ordinary coherence with a reduction in the deg

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[i] Throughout the paper, frequencies are measured in units of inverse time. Multiplying these frequencies by 2π converts them to radian measure.

[ii] This notation means that $\int_{f_1}^{f_2} P_{XY}(f)df = E[(Z_X(f_2) - Z_X(f_1))\overline{(Z_Y(f_2) - Z_Y(f_1))}]$.

[iii] Following Beauchamp and Yuen (1979), L should be approximately $N/16$.

[iv] The latter adjustment will remove non-stationarity due to the existence of a purely deterministic waveform in the data, see Hinich and Wild (1999).

[v] See Beauchamp and Yuen (1979) and Yuen and Fraser (1979) for more details on the alternative procedure.

[vi] Some authors have used alternative confidence intervals for coherence (and phase angle). See McCullough (1995) for one example.