Question 1 – A credit problem Consider the firm problem entrepreneurs considering borrowing to finance their operations, and the problem of households who consider lending to the firms. The timing is as follows:

At $t = 0$, firms start with 0 capital. They can issue bonds b at a bond price of q in order to borrow. They use these funds to purchase machinery k. There are no other options of financing investment; and there are no other uses for the borrowed funds – therefore the amount borrowed equals the amount of capital purchased. On the other side of the loan market there is a household that can buy the bonds in order to save for next period. That household receives a fixed endowment e_0 in period 0, which they can split up between saving and consumption.

At $t = 1$ firms produce with the capital stock they have acquired, according to the production function zk^{α}. Productivity z is a random variable realized at $t = 1$ before production occurs. This means the realization of z is unknown at time $t = 0$ when the borrowing and investment choice is made (however the distribution of z is known at $t = 0$). After production the firm sells the output goods, sells the remaining capital stock $(1 - \delta) k$ and repays the bond holders. The household does not receive any endowments in period 1, instead they only receive the bond repayments and consume them.

Households' preferences over the two periods are given by $u(c_0) + \beta E[u(c_1)]$. Firms are risk-neutral and maximize expected profits.

1. Formulate the firm's problem. Derive the first-order condition.
2. Formulate the household's problem, including possible inequality constraints. Then, assuming the inequality constraints do not bind, derive the first-order necessary condition for an interior solution.

For the remainder of the problem, assume full depreciation and let's say for simplicity that the household is risk-neutral, i.e. that u is linear. (Note that this means that an interior solution here is a non-generic case; however we can use the FOC because it could be derived as a non-arbitrage condition in a fuller model.)

3. Assume that $z = 1$ with certainty. What is the equilibrium capital stock in terms of parameters?
4. Now assume that $z \in \{0, 2\}$, and each realization happens with 50% probability. Compared to the situation from the previous part, what is the effect of this mean-preserving spread on investment, the bond price, on realized firm profits, and on household welfare? Does the increase in risk affect households' welfare negatively? If so, why? If not, why not?

Now consider a limited liability situation:

At $t = 0$ firms again borrow from households to finance investment, just as before.

At $t = 1$, however, the firm now first considers its revenue from sales of output zk^{α} before repaying the bondholders, these are the firm's available funds. In particular, if this amount is greater than the outstanding debt, the firm repays the bonds in full. But if available funds are less than the outstanding debt, the firm defaults and only repays the amount of available funds.[1] This structure is known to all agents at time 0.

5. Set up the modified firm problem and household problem under limited liability. State the first-order conditions for an interior solution (note: you do not have to solve them).
6. Now assume again that $z \in \{0, 2\}$ with each realization happening with 50% probability. Compare the equilibrium bond price and the amount of investment to the situation from 4. and comment. Is household welfare affected negatively this time? If so, why? If not, why not?

[1] In other words, flow profits in this situation can not be negative. This means that firm owners are not liable for the firm's debt; a situation called "limited liability". It applies, for example, to public companies where ownership is hold by a firm's stockholders. The stockholders are not liable to pay out the firm's creditors in case the firm defaults.

Question 1 – A credit problem Solutions:

1. The firm's problem is to

$$\max_{b,k} E\left[zk^{\alpha}+(1-\delta)k-b\right] \text{ s.t. } k=qb$$

or

$$\max_{k} E\left[zk^{\alpha}+(1-\delta)k-k/q\right].$$

The first-order condition is

$$\begin{aligned}\frac{1}{q}-(1-\delta)&=E\left[z\alpha k^{\alpha-1}\right]\\ \frac{1}{q}-(1-\delta)&=\alpha k^{\alpha-1}E[z]\end{aligned}$$

2. Since in this setup the households are guarantueed payback of the bonds, the expectation over period-1 utility is trivial. The household's problem is to

$$\begin{aligned}\max_{c_0,c_1,b}\ & u(c_0)+\beta u(c_1)\\ \text{s.t. } c_0&=e_0-qb\\ c_1&=b\\ c_0&\geq 0\\ c_1&\geq 0\end{aligned}$$

or

$$\begin{aligned}\max_{b} u(e_0-qb)+\beta u(b)+\lambda_0(e_0-qb)+\lambda_1 b&\\ \text{s.t. } \lambda_0 c_0&=0\\ \lambda_1 c_1&=0\end{aligned}$$

The first-order condition for an interior solution (i.e. $\lambda_0=\lambda_1=0$) is

$$\begin{aligned}u'(c_0)q&=\beta u'(c_1)\\ q&=\beta\frac{u'(c_1)}{u'(c_0)}.\end{aligned}$$

3. With linear utility, the first-order condition becomes $q=\beta$. From firms' first-order condition we have

$$\begin{aligned}k^{*}&=\left(\frac{\alpha E[z]}{\frac{1}{q}-(1-\delta)}\right)^{\frac{1}{1-\alpha}}=\left(\frac{q\alpha E[z]}{1-q(1-\delta)}\right)^{\frac{1}{1-\alpha}}\\ &=\left(\frac{\beta\alpha E[z]}{1-\beta(1-\delta)}\right)^{\frac{1}{1-\alpha}}\end{aligned}$$

with the equilibrium bond price substituted in.

4. From the Hhs FOC we know that the bond price is constant. We can see from the optimal investment condition in 3. that a mean-preserving spread does not affect optimal investment (at least if firm and households are both risk-neutral as we assumed). While the firm's *expected* profits are not affected, the firm's realized profits now depend on the realization of z. If $z=0$ the firm is running a loss with realized profits $\pi=k[1-\delta-1/\beta]$. Because firms always repay the bond, household welfare is not affected.

5. First note that now repayments are $\min\{zk^\alpha + (1-\delta)k, b\}$. Since households now know that there is a risk of default, the problem becomes

$$\max_b u(e_0 - qb) + \beta E\left[u\left(\min\{zk^\alpha + (1-\delta)k, b\}\right)\right]$$

with FOC

$$\begin{aligned} u'(c_0)q &= \beta dE\left[u\left(\min\{zk^\alpha + (1-\delta)k, b\}\right)\right]/db \\ q &= \beta\frac{d}{db}E\left[\min\{zk^\alpha + (1-\delta)k, b\}\right] \end{aligned}$$

using linear utility.
Again using $b = k/q$, for the firm the modified problem is to

$$\max_k E\left[\max\{zk^\alpha + (1-\delta)k - k/q, 0\}\right]$$

FOC:

$$\frac{d}{dk}E\left[\max\{zk^\alpha + (1-\delta)k - k/q, 0\}\right] = 0$$

6. Using the hint, let's guess that

$$\min\{zk^\alpha + (1-\delta)k, b\} = \begin{cases} (1-\delta)k & \text{if } z = 0 \\ b & \text{if } z = 2 \end{cases}.$$

Then the Hh FOC becomes

$$\begin{aligned} q &= \beta\frac{d}{db}\left[\frac{1}{2}(1-\delta)k + \frac{1}{2}b\right] \\ &= \frac{1}{2}\beta. \end{aligned}$$

The firm FOC is

$$\begin{aligned} \frac{d}{dk}\left[\frac{1}{2}\left(z_H k^\alpha + (1-\delta)k - k/q\right) + \frac{1}{2}0\right] &= 0 \\ z_H \alpha k^{\alpha-1} &= \frac{1}{q}. \end{aligned}$$

Combining the FOCs,

$$\begin{aligned} q\alpha k^{\alpha-1} &= 1 \\ \beta\alpha &= k^{1-\alpha} \\ k &= (\alpha\beta)^{\frac{1}{1-\alpha}}. \end{aligned}$$

In other words, limited liability on its own does **not** necessarily create inefficient allocations (however, it can do so under small modifications, for example by introducing default costs).

Question 2 – An optimal timing problem A young couple has recently graduated and entered the job market. They currently rent an apartment, but their ultimate goal is to build a house. Both have steady jobs earning them a combined wage income of constant real wage $w_t = w$ every period, regardless of their living situation. Rent costs r goods per period, and they do not borrow nor save. Both are talented craftspeople, so the couple have the option of constructing a house – however building a house costs a lot of effort and therefore causes disutility of ϕ in the period that it is built (for simplicity let's abstract from the cost of the materials). If they do decide to construct a house, then the timing is as follows: the house is built in the current period (while they still rent), and the move-in date is before the beginning of next period. Once the couple own their home they stay there forever and receive extra utility of ν every period.

Like any other representative household they also receive utility from consumption of $u(c_t)$ and have a discount factor of β. The couple plan for an infinite horizon.

1. What are the state and control variables for the household? Write down the household's problem recursively.

2. Derive the household's decision rule for building a house, i.e. derive a condition indicating whether it is optimal to construct a house or not. What are the dynamics of this decision rule (i.e. what does it tell us about the timing of the decision)?

Now assume that instead of being constant, the utility cost of construction ϕ is a random variable, for which every period a realization ϕ_i is drawn i.i.d. from a known distribution F. (For example, building a house in a bad weather season is much less pleasant than during good weather.)

3. What are the state and control variables for the household now? Write down the modified household's problem recursively.

4. Derive the household's decision rule for building a house as a function of parameters. To do this, guess and verify that there exists a cutoff value $\bar{\phi}$ such that the couple build a house as soon as they draw a value for $\phi < \bar{\phi}$ (you will need this guess to write out the expectation of the continuation value).

5. Bonus question (if you have found the decision rule): Use your decision rule to show that if the couple is very impatient ($\beta \to 0$) they always rent, whereas if they are very patient ($\beta \to 1$) they will always build.

Solutions:

1. The only state variable is the current living situation, i.e. a binary{rent, own}. The only choice is next period's living arrangement – and because the household never goes from owning back to renting this really only applies to the renting situation. (We could include consumption as a choice variable which would be trivial since the budget constraint is just $c = w - r$.) With the description of the timing, the household problem can be expressed as

$$V^r = u(w-r) + \max\{\beta V^r, -\phi + \beta V^o\}$$
$$V^o = u(w) + \nu + \beta V^o.$$

2. The question is, under which condition is the household going to choose renting over building & owning? That is, we are looking for the combinations of paramters such that

$$\beta V^r > -\phi + \beta V^o.$$

The first thing to note that if this condition is true in this period, then it is true for all periods (because renting/owning is the only state variable, so both sides of the inequality are constant). This means that we can already say something about the dynamics of the decision rule: Either the couple build a house right away, or never at all. We can therefore also write $V^r = u(w-r) + \beta V^r \Rightarrow V^r = u(w-r)/(1-\beta)$. For the value of owning we get $V^o = \frac{u(w)+\nu}{1-\beta}$. Substituting in above, we have

$$\beta \frac{u(w-r)}{1-\beta} > -\phi + \beta \frac{u(w)+\nu}{1-\beta}$$
$$\phi > \beta \frac{u(w) - u(w-r) + \nu}{1-\beta}.$$

Intuitively, this is simply comparing the costs and benefits of building a house vs renting: The cost ϕ is due today. The benefits are the increase in utility from consumption due to saved rent $u(w) - u(w-r)$ as well as the extra utility from living in their own place ν. These are utility flows that last indefinitely (therefore multiplied with $1/(1-\beta)$), but only kick in with a 1 period delay (factor β). Of course this means that the higher the rent r and the higher the utility from owning ν the more likely the household is to build the house, whereas the greater the construction cost ϕ and the more impatient the household (lower β), the more likely they are to rent.

3. Now the construction cost ϕ is variable rather than a constant parameter - it therefore needs to be included as a state variable to the problem (although again this matters only for the renting state). Additionally, since the value of ϕ varies randomly from period to period, the household needs to form expectations about the future realizations of ϕ. The modified value for renting is

$$V^r(\phi) = u(w-r) + \max\{\beta E[V^r(\phi')], -\phi + \beta V^o\},$$

whereas the value for owning remains the same as before.

4. We are again looking for

$$\beta E[V^r(\phi')] > -\phi + \beta V^o.$$

This time, while the left hand side of the inequality $\beta E[V^r(\phi')]$ is constant, the right hand side is not because ϕ fluctuates over time. The expectation is

$$\begin{aligned}
E\left[V^r\left(\phi\right)\right] &= E\left[u\left(w-r\right)+\max\left\{\beta E\left[V^r\left(\phi'\right)\right], -\phi+\beta V^o\right\}\right] \\
&= u\left(w-r\right)+E\left[\max\left\{\beta E\left[V^r\left(\phi'\right)\right], -\phi+\beta V^o\right\}\right] \\
&= u\left(w-r\right)+\int_0^{\bar{\phi}} -\phi d\phi+\beta V^o+\int_{\bar{\phi}}^{\infty} \beta E\left[V^r\left(\phi'\right)\right] d\phi \\
&= u\left(w-r\right)+\int_0^{\bar{\phi}} -\phi d\phi+\beta V^o+\left[1-F\left(\bar{\phi}\right)\right] \beta E\left[V^r\left(\phi\right)\right] \\
E\left[V^r\left(\phi\right)\right]\left(1-\beta\left[1-F\left(\bar{\phi}\right)\right]\right) &= u\left(w-r\right)+\int_0^{\bar{\phi}} -\phi d\phi+\beta V^o \\
E\left[V^r\left(\phi\right)\right] &= \frac{u\left(w-r\right)+\int_0^{\bar{\phi}} -\phi d\phi+\beta V^o}{1-\beta\left[1-F\left(\bar{\phi}\right)\right]}.
\end{aligned}$$

Substituting this into the above gives

$$\begin{aligned}
\beta \frac{u\left(w-r\right)+\int_0^{\bar{\phi}} -\phi d\phi+\beta V^o}{1-\beta\left[1-F\left(\bar{\phi}\right)\right]} &> -\frac{\phi}{\beta}+V^o \\
\beta u\left(w-r\right)+\beta \int_0^{\bar{\phi}} -\phi d\phi+\beta^2 V^o &> \left(1-\beta\left[1-F\left(\bar{\phi}\right)\right]\right)\left(-\frac{\phi}{\beta}+V^o\right) \\
\beta u\left(w-r\right)+\beta \int_0^{\bar{\phi}} -\phi d\phi+\left(1-\beta\left[1-F\left(\bar{\phi}\right)\right]\right)\frac{\phi}{\beta} &> V^o\left(1-\beta\left[1-F\left(\bar{\phi}\right)\right]+\beta^2\right) \\
\beta u\left(w-r\right)+\beta \int_0^{\bar{\phi}} -\phi d\phi+\left(1-\beta\left[1-F\left(\bar{\phi}\right)\right]\right)\frac{\phi}{\beta} &> \frac{u\left(w\right)+\nu}{1-\beta}\left(1-\beta\left[1-F\left(\bar{\phi}\right)\right]+\beta^2\right).
\end{aligned}$$

Note that these are all constants except for ϕ which is the random variable. The condition implicitly defines a $\bar{\phi}$ such that if $\phi \leq \bar{\phi}$ the inequality no longer holds (the r.h.s. becomes less or equal than the l.h.s.).

5. We can see that if the household is very impatient the utility cost becomes very important as $\phi/\beta \to \infty$, and the condition will hold for sure (and household rents). Conversely, if the household becomes very patient then the value of owning a home $V^o = \frac{u(w)+\nu}{1-\beta}$ dominates, makes the r.h.s. arbitrarily large so that the condition fails and Hhs build for sure.