PRIMAL AND DUAL REPRESENTATIONS OF THE ERROR IN SUMMATION MONETARY AGGREGATES

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Abstract: We provide primal and dual representations for the tracking error in the official monetary aggregates published by the Federal Reserve System from the standpoint of monetary aggregation theory. We show that the tracking error will be zero if either all asset stocks or their prices change proportionally. This result implies that the squared tracking error is a function of the degree of non-proportionality in the changes in either asset stocks or their prices. We present two measures of the degree of non-proportionality that were originally developed by Theil (1967) and Allen and Diewert (1981). We first illustrate the interaction between the tracking error and these measures using simulated data. We next examine the available time series, which indicate that the degree of non-proportionality in monetary data changes over time and may increase systematically under certain monetary policy regimes. In particular, we show that the degree of non-proportionality for both asset stocks and their prices was very high during 1978-1982, when the Federal Reserve was targeting nominal monetary aggregates, contributing to a large error in the measurement of the underlying policy instrument.

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The monetary quantities aggregates produced by the Federal Reserve System, such as M1, M2, and M3, are summations of the stocks of currency and deposits in the aggregate. Although, these summation aggregates have been widely used in empirical research, they have been criticized because they are not valid economic aggregates from the standpoint of economic aggregation theory. In aggregation theory, the economic quantity aggregate for a weakly separable group of goods and services is the indirect utility function and the dual price aggregate is the expenditure function.\footnote{Barnett (1978, 1980) introduced the aggregation theory into monetary economics.} Monetary aggregation theory treats monetary assets as weakly separable groups of durable goods. The user cost of a monetary asset is the foregone interest income incurred as a result of holding the monetary assets.\footnote{Statistical index number methods can be used to measure economic quantity and price aggregates for monetary assets from user cost and asset stock data.} The summation aggregate coincides with the economic quantity aggregate only under strong assumptions about the substitutability of monetary assets, which have counterfactual implications. The summation aggregate is also not known to have any desirable statistical properties if these assumptions are violated. There is a substantial body of empirical research, which has generally concluded that superlative index number approximations to the economic aggregates outperform summation aggregates.\footnote{We provide theoretical results on the relationship between the summation aggregate and the economic quantity aggregate. We regard the summation aggregate as an approximation to the economic quantity aggregate, and derive two alternative representations of its tracking error. The primal representation shows that the tracking error is a function of the changes in the component asset stocks (quantities), and the dual representation shows that the error is a function of changes in component user costs (prices). We show that the tracking error will be zero if either all prices or quantities change proportionally, and that the tracking error is a function of the degree of non-proportionality in the changes in either prices or quantities. We present two measures of the degree of non-proportionality developed by Theil (1967) and Allen and Diewert (1981). Using simulated data, we show that these measures of the degree of non-proportional price and quantity change are highly correlated with the squared tracking error of the summation aggregate. These theoretical results have important empirical implications, because the available evidence indicates that the degree of non-proportionality changes over time and may increase systematically under certain monetary policy regimes. In particular, we show that the degree of}
non-proportionality of changes in both user cost prices and asset stocks was very high during 1978-1982, when the Federal Reserve was targeting non-borrowed reserves. The non-borrowed reserves targets were associated with greater interest rate volatility, and, consequently, were also associated with a high degree of non-proportional price change. We show that this led to large tracking errors in the summation M1 and M2 aggregates. This observation could be used to explain the breakdown of certain empirical targeting relationships over this same period.6 We also show that the degree of non-proportional changes in user cost prices and asset stocks was very high for summation M1, but not for M2, during the period 1972-1976. This led to large tracking errors for summation M1 that could help to explain the famous “case of the missing money” in this period investigated by Goldfeld (1976).

The possibility that there is a systematic relationship between the measurement error in a summation monetary aggregate and changes in monetary policy is potentially very damaging to empirical evidence based on summation aggregates. The idea that measurement error can obscure money’s relevance to other macroeconomic variables was referred to as the “Barnett critique” by Chrystal and MacDonald (1994, pp.76). We argue that the Barnett critique has similar importance for empirical analysis as the “Lucas critique.” Lucas (1973) stressed that the interaction between policy events and expectations formation could be critical in evaluating empirical evidence. We show that the interaction between monetary policy events and measurement error could be critical in evaluating empirical monetary evidence.

This paper is organized as follows. In Section 1, we provide some necessary background on economic aggregation theory for monetary assets. In Section 2, we provide primal and dual representations of the tracking error of summation aggregates. In Section 3, we provide simulation results. In Section 4, we provide empirical analysis and suggest some policy implications. In Section 5, we conclude.

I. Economic Approach to Monetary Aggregation

The economic approach to monetary aggregation proceeds from a microeconomic decision problem.7 For exposition, we base our discussion on the utility maximization problem for a household. Barnett (1987) provides extensions to other cases. The household is assumed to maximize at every time \( t \) a homothetic time-invariant utility function, \( U \), subject to a budget constraint:
\[
\max_m \{ U(m) : \langle \pi, m \rangle = y \}, \tag{1}
\]

where \( m \) is an \( N \)-dimensional vector of monetary assets, \( \pi \) is an \( N \)-dimensional vector of user costs, \( y \) is total expenditure allocated to the monetary portfolio, and \( \langle \pi, m \rangle = \sum_{i=1}^{N} \pi_i m_i \). We use the notation \( m \) to refer to the \( \arg \max \{ U(m) : \langle \pi, m \rangle = y \} \). The utility function \( U \) is called an aggregator function and is assumed to satisfy the following properties: i) continuity; ii) monotonicity; iii) quasi-concavity; and iv) homotheticity. If \( m \) represents a symmetrically homothetically weakly separable group in a more complex intertemporal, utility maximization problem, then \( y \) is the total expenditure allocated to the \( N \) assets at time \( t \) implied by the optimal solution to the more complex problem, and \( U \) is the category sub-utility function for the weakly separable group of assets. Under mild regularity conditions on the utility function for the more complex decision, the optimal solution from (1) will coincide with the optimal solution for the \( N \) assets from the full decision.\(^8\)

Barnett (1978, 1980, 1987) treated monetary assets as durable goods, and proved that the nominal user cost for monetary asset \( i \) at time \( t \) is

\[
\pi_i = p_i^* \frac{R_i - r_u}{1 + R_i}, \tag{2}
\]

where, \( r_u \) is the holding period yield on asset \( i \) in period \( t \), \( p_i^* \) is the value of a true cost of living index in period \( t \), and \( R_i \) is the risk-free rate of return on an asset that can be used to intertemporally transfer wealth, but does not provide any services to the agent, except possibly in the final period of the planning horizon.\(^9\) This asset is called the benchmark asset and it is assumed to dominate all other assets as a risk-free store of value. The user cost of monetary asset \( i \) is the foregone interest \((R_i - r_u)\) paid on one dollar of nominal deposits at the end of the holding period, discounted to present value and represents the (shadow) price of the asset. To simplify exposition, we will simply refer to \( \pi_i \) as the price of monetary asset \( i \) at time \( t \). The values of the relative prices are equal to \((R_i - r_u)/(R_i - r_u)\).

Economic aggregation theory defines quantity aggregates that are functions of component quantities, and price aggregates that are functions of prices. If \( U \) satisfies properties i) - iv) then the quantity aggregate at time \( t \) is the indirect utility, \( U(m_t) \), and the price aggregate at time \( t \) is the
unit expenditure function, \( e(\pi_t,1) = \min_h \{<\pi_t,h>: U(h) = 1\} \). The quantity and price aggregates are positive linearly homogeneous in their respective arguments, which implies that a proportional change in all components of \( m_t \) or \( \pi_t \) will result in the same proportional change in the corresponding aggregate. The quantity and price aggregates also satisfy strong factor reversal, meaning that \( e(\pi_t,1) \cdot U(m_t) = y_t \) for any time \( t \).

The exact Malmquist (1953) quantity index \( M(m_1,m_0) = U(m_1)/U(m_0) \), is the quantity aggregate normalized on a base period, where \( m_1 \) is the optimal asset vector in period 1, and \( m_0 \) is the optimal quantity vector in the base period. The exact Konüs (1939) price index \( K(\pi_1,\pi_0) = e(\pi_1,1)/e(\pi_0,1) \) is the price aggregate normalized on a base period, where \( \pi_1 \) is the user cost vector in period 1, and \( \pi_0 \) is the user cost vector in the base period. These indexes are said to be exact because they are equal to the change of the underlying economic aggregate relative to a base period for all price and quantity vectors. Exactness is defined in relative terms because the utility function is ordinal. The exact indexes satisfy weak factor reversal, meaning that \( M(m_1,m_0) \cdot K(\pi_1,\pi_0) = y_1/y_0 \) for any two periods. In the homothetic case, strong and weak factor reversal are equivalent.

These results provide the basic properties of quantity and price aggregates: (i) The price (quantity) aggregate is a function only of prices (quantities); (ii) The price (quantity) aggregate is positive linearly homogeneous in prices (quantities); and (iii) The price and quantity aggregates satisfy weak or strong factor reversal.

In continuous time, the homothetic category sub-utility function satisfies the following equation:

\[
d \ln(U(m_t)) = \sum_{i=1}^{N} s_{it} d \ln(m_{it}) ,
\]

where \( s_{it} = \pi_{it} m_{it} / \sum_{j=1}^{N} \pi_{jt} m_{jt} \) is the expenditure share for asset \( i \). The solution to the differential equation is a Divisia index.

II. Theoretical Error in Summation Aggregate

In this section, we regard the summation aggregate as an approximation to the underlying economic quantity aggregate and present a theorem describing the approximation error in the summation aggregate. This theorem allows us to state a new formulation of the tracking error in
summations aggregates, which is useful in analyzing the source of the tracking error. Before presenting these results, we discuss cases where the summation aggregate is exact and the related issue of defining a price aggregate that is dual to the summation aggregate.

II.1 Exactness of the Summation Aggregate in Two Special Cases

The economic quantity aggregate is a function of quantities, and the economic price aggregate is a function of prices. The summation aggregate attempts to measure the quantity aggregate by taking a summation of the stocks of the monetary assets. We define the summation aggregate, \( S \), as

\[
S(m_i) = \sum_{i=1}^{N} m_{it}.
\]

The summation aggregate is exact if \( S(m_t)/S(m_0) = M(m_t, m_0) \) holds at every time \( t \) for any base period. Although the summation aggregate is not known to have any desirable statistical properties, it is exact in two cases. We discuss these two cases and their respective implications for the price aggregate dual to the summation quantity aggregate.

First, the summation aggregate is exact if the category sub-utility function \( U \) is linear with identical coefficients on all monetary assets, so that

\[
U(m) = a \sum_{i=1}^{N} m_i.
\]

In this case, the summation aggregate is a particular cardinalization of the microeconomic quantity aggregate. The linear indifference curves associated with such a utility function imply that the optimal solution is on a corner, unless all prices are identical. The household holds only one monetary asset -- the asset with the lowest price. The dual price aggregate would therefore be the minimum price,

\[
P^L_t = \min_h \left\{ \langle \pi_t, h \rangle : U(h) = \sum_{i=1}^{N} h_i = 1 \right\} = \min\{\pi_i\}.
\]

This dual price aggregate will, however, generally strong factor reverse with the summation aggregate only if the solution is a corner. This condition is counterfactual, however, because households typically hold a portfolio of assets that have different interest rates (and thus different user costs).

Second, the summation aggregate is exact if utility is Leontief

\[
U(m) = \min\{a_1m_1, ..., a_Sm_S\}.
\]
At the optimum, $a_1m_1^* = ... = a_Nm_N^*$, and this implies that $U(m) = a_1m_1^* = ... = a_Nm_N^*$. The summation aggregate is $S(m^*) = m_1^* + m_2^* + m_3^* = m_1^*(1 + \frac{a_1}{a_2} + ... + \frac{a_1}{a_N})$, which is again a particular cardinalization of the microeconomic quantity aggregate. The dual price aggregate would be a weighted summation of the user cost prices,

$$P_i^s = \min_h \{<\pi_i, h> : U(h) = \min\{\alpha_i h_i \}_{i=1}^N = 1\} = \sum_{i=1}^N \frac{\pi_i}{\alpha_i}$$

(6)

This dual price aggregate will generally factor reverse with the summation aggregate only if the quantities are always proportional, which is again counterfactual.

Both of these exact cases are very restrictive. We can also examine a third method of defining a dual price aggregate for the summation aggregate. This method ignores the microeconomic criteria of exactness and instead uses the property of factor reversal to define the dual price aggregate. Moore, Porter, and Small (1990) define the own rate of summation M2 as the deposit weighted average of deposit rates on the components of the M2 monetary aggregate, and define the opportunity cost of M2 as the difference between a competing interest rate (their suggestion is the 3 month treasury bill rate) and the own rate of M2. We can define the interest rate on the summation aggregate as $r^s = <r, w>$, where $w_i = m_i / S(m) = m_i / \sum_{j=1}^N m_j$. The corresponding price aggregate is defined as

$$P_{i, FR} = p_i^s \frac{R_i - r^s}{1 + R_i}$$

(7)

which resembles the user cost price for a single monetary asset. It can be shown that $S(m_i)$ and $P_{i, FR}$ satisfy factor reversal:

$$P_{i, FR} \cdot S(m_i) = p_i^s \frac{R_i - r^s}{1 + R_i} \sum_{j=1}^N m_{ij} = \sum_{i=1}^N p_i^s (R_i - r_i) m_{ij} = <m_i, \pi_i> = y_i$$

(8)

This definition of a price aggregate for the summation aggregate is not universally acceptable, even though it is based on strong factor reversal, because it is inconsistent with economic aggregation theory unless the summation aggregate is actually exact.

The fact that the duality of the price and quantity aggregates can only be achieved by sacrificing the factor reversal property is indicative of the underlying problem with the summation
aggregate. Namely, that it is based on either the assumption that the assets are perfect substitutes or that the utility function is Leontief, both of which have counterfactual implications.

II.2 Primal and Dual Representations of the Error in the Summation Aggregate

The restrictive nature of the exact cases implies that the summation aggregate will not generally be exact. We define the approximation error in the summation aggregate as \( \varepsilon_t = S(m_t) / U(m_t) \). The utility function can be monotonically transformed, so we must consider a measure of error that is invariant to monotonic transforms. In continuous time, we define the tracking error as the percentage change in the approximation error:

\[
\frac{d \ln(\varepsilon_t)}{d \ln(S(m_t))} = d \ln(S(m_t)) - d \ln(U(m_t))
\]

We focus on this tracking error. The homothetic utility function, or any monotonic transformation of it, satisfies the Divisia differential equation:

\[
d \ln(U(m_t)) = \sum_{i=1}^{N} s_i d \ln(m_{it}).
\]

The tracking error may be zero even if the summation aggregate is not exact. The summation aggregate satisfies the differential equation:

\[
d \ln(S(M_t)) = \sum_{i=1}^{N} w_i d \ln(m_{it}),
\]

where \( w_i = m_i / S(m) = m_i / \sum_{j=1}^{N} m_j \). Thus, the tracking error satisfies the differential equation:

\[
d \ln(\varepsilon_t) = \sum_{i=1}^{N} w_i d \ln(m_{it}) - \sum_{i=1}^{N} s_i d \ln(m_{it})
= \sum_{i=1}^{N} (w_i - s_i) d \ln(m_{it})
\]

If the monetary asset stocks change proportionally in a given period then \( d \ln(m_{it}) = d \ln(m_{it}) = k \) for all \( i,j \) and the tracking error is \( k \cdot \sum_{i=1}^{N} (w_i - s_i) = k \cdot \left( \sum_{i=1}^{N} w_i - \sum_{i=1}^{N} s_i \right) = 0 \), because both sets of weights sum to one. This result is numerical and does not require that the summation aggregate be exact. Nevertheless, it is intuitive, because we have already shown that the summation aggregate is exact in the case of Leontief preferences, which implies that at the optimum changes in the asset stocks are always proportional. This result shows that even if quantities do not always change proportionally (as they would in the Leontief case), the tracking error will be zero when quantities do change proportionally.
Equation (10) can be rewritten and interpreted further. The weights, \((w_i - s_i)\), sum to 1, which allows us to normalize the changes in asset stocks using the quantity aggregate as follows:

\[
d \ln(\varepsilon_i) = \sum_{i=1}^{N} (w_{ii} - s_{ii}) \left( d \ln(m_{ii}) - d \ln(U(m_i)) \right) \quad \text{[Primal]}
\]

We call this the **primal representation** of the tracking error. The primal representation is known in index number theory, and can be easily approximated in discrete time using available monetary data (see Section IV of this paper). The squared tracking error is

\[
(d \ln(\varepsilon_i))^2 = \left( \sum_{i=1}^{N} (w_{ii} - s_{ii}) \left( d \ln(m_{ii}) - d \ln(U(m_i)) \right) \right)^2.
\] (11)

This formula implies that larger non-proportional changes in the asset stocks will lead to larger squared tracking errors in the summation aggregate. This representation is very useful, because it provides a numerical estimate of the magnitude of the error in the summation aggregate.

The dual representation of the tracking error can be derived using standard microeconomic duality theorems and linear homogeneity. We must first establish a dual representation of the approximation error:

**Theorem:** Dual representation of the summation approximation error

Let \(m_i\) denote the N-dimensional vector of Marshallian demand functions, \(m_i = m(\pi_i, y_i)\), for the utility maximization problem in (1), let \(h_i = h(\pi_i, u)\) denote the vector of Hicksian demand functions, and let \(U\) satisfy i) – iv). Then the following relationships hold:

i) \(S(m_i) = U(m_i) \varepsilon_i\); and

ii) \(P_i^{FR} = e(\pi_i, 1) / \varepsilon_i\),

where \(\varepsilon_i = \sum_{i=1}^{N} h_i(\pi_i, 1)\).

**Proof:** The standard duality theorems imply that \(m_i(\pi_i, y_i) = h_i(\pi_i, U(m(\pi_i, y_i)))\) for all \(i = 1, \ldots, N\), implying that \(S(m_i) = \sum_{i=1}^{N} m_i(\pi_i, y_i) = \sum_{i=1}^{N} h_i(\pi_i, U(m(\pi_i, y_i)))\). Linear homogeneity implies that \(\sum_{i=1}^{N} h_i(\pi_i, U(m(\pi_i, y_i))) = \sum_{i=1}^{N} h_i(\pi_i, 1) U(m(\pi_i, y_i)) = U(m_i) \sum_{i=1}^{N} h_i(\pi_i, 1)\).

The second result is derived from the fact that \(P_i^{FR}\) factor reverses with the summation quantity aggregate, so that \(P_i^{FR} \cdot S(m_i) = e(\pi_i, m_i) \varepsilon_i\). Optimization and linear homogeneity imply that \(y = \pi \cdot m = e(\pi, 1) U(m)\), and therefore, \(P_i^{FR} = e(\pi_i, 1) \cdot U(m_i) / S(m_i) = e(\pi_i, 1) / \varepsilon_i\). \(\square\)
We derive the dual form of the tracking error by log differentiating $\epsilon_t = \sum_{i=1}^{N} h_i (\pi_t, 1)$. The dual form is

$$d \ln(\epsilon_t) = \sum_{i=1}^{N} \eta_i \cdot d \ln(\pi_t)$$

where the weights, $\eta_i = \sum_{j=1}^{N} \frac{\partial h_j (\pi, 1)}{\partial \pi_t} \cdot \frac{\pi_t}{\varepsilon}$, satisfy the property $\sum_{i=1}^{N} \eta_i = 0$. If all prices increase by the same percentage (i.e. change proportionally) then the tracking error will be zero. We can normalize all user cost prices using the price of some numeraire, $\pi_t^{Num}$ to produce the dual representation of the tracking error,

$$d \ln(\epsilon_t) = \sum_{i=1}^{N} \eta_i \cdot [d \ln(\pi_t) - d \ln(\pi_t^{Num})] \quad \text{[Dual]}$$

which is a function of the percentage change in relative prices. Intuitively, if all prices change proportionally, then relative prices do not change and there should not be any substitution between assets in that period. But, if there is no substitution between the assets, then the tracking error should be zero in that period. The squared tracking error is

$$(d \ln(\epsilon_t))^2 = \left(\sum_{i=1}^{N} \eta_i \cdot [d \ln(\pi_t) - d \ln(\pi_t^{Num})]\right)^2.$$

This formula implies that larger non-proportional changes in relative prices lead to larger squared tracking errors in the summation aggregate.

To summarize, the primal and dual representations of the tracking error show that if either all quantities or all prices change proportionally then the tracking error is zero. This implies that the summation aggregate can have large tracking errors in some periods and small tracking errors in other periods. If we disregard the special cases under which the summation aggregate is exact, then the primal and dual representations provide the basis for interpreting the empirical performance of summation aggregates. In particular, measures of the degree of non-proportionality become critical to evaluating the summation aggregate.

### III. Measuring the Degree of Non-Proportionality in Price and Quantity Change

In this section, we propose some measures of the degree of non-proportionality of price and quantity change and study the correlation between these measures and the tracking error for simulated monetary data.
III.1 Measuring the Degree of Non-Proportional Price and Quantity Change

The dual representation theorem implies that the squared tracking error, $(d \ln(\varepsilon_t))^2$, is larger during periods of large non-proportional price or quantity changes (i.e. large changes in relative prices or quantities). There are numerous ways to measure the amount of non-proportional price change. Intuitively, a measure of the degree of non-proportional price change would aggregate terms of the form: $(d \ln(\pi_i) - d \ln(\pi_i^{Num}))^2$.

Allen and Diewert (1981) suggested a measure of the “deviation from proportionality” of prices by regressing the percentage change in each component price against a constant, and use the sum of squared residuals from the regression as a measure of non-proportional price change. We use the average sum of squared residuals:

$$J_{tAD} = \frac{1}{N} \sum_{i=1}^{N} \left( d \ln(\pi_i) - \left( \sum_{j=1}^{N} s_{ij} \cdot d \ln(\pi_j) / N \right) \right)^2. \quad (14)$$

Averaging the sum of squared residuals can be used to correct for changes in the number of components and is consistent with the formulation in Diewert (1995a). In this formula, prices are normalized using the arithmetic mean of the component prices. The measure is the average of the squared percentage changes in normalized prices.

A second measure would be to normalize the user cost prices by a Divisia price index and weight the percentage change in normalized user cost prices by the expenditure shares. This would produce the formula:

$$J_{tT} = \sum_{i=1}^{N} s_{it} \cdot \left( d \ln(\pi_i) - \left( \sum_{j=1}^{N} s_{ij} \cdot d \ln(\pi_j) \right) \right)^2. \quad (15)$$

This is a continuous time version of the Divisia price second moment, defined by Theil (1967) and investigated by Barnett and Serletis (1990) and Barnett, Jones, and Nesmith (1996).

The connection between these two measures and the tracking error is highlighted by the following formula:

$$(d \ln(\varepsilon_t))^2 = \left( \sum_{i=1}^{N} \eta_{it} \cdot \left[ d \ln(\pi_i) - d \ln(\pi_i^{Num}) \right] \right)^2$$

\[= \sum_{i=1}^{N} \eta_{it}^2 \cdot \left( d \ln(\pi_i) - d \ln(\pi_i^{Num}) \right)^2 + \text{cross terms} \quad (16)$$

If you neglect the cross terms then $J_{tAD}^T$ and $J_{tT}^T$ represent guesses about the value of the squared weights, which are themselves functions of the prices, and use different normalizations to produce
relative user cost prices. The Allen-Diewert measure normalizes prices using the arithmetic average of the relative price changes and weights all price changes equally, whereas the Divisia second moment normalizes prices using the Divisia price index, and weights the price changes by their expenditure shares. We call these measures as the Allen-Diewert and Theil price variances.

The primal representation shows that the tracking error can be viewed as a function of the degree of non-proportional asset stock (quantity) change. We suggest the following two measures of the degree of non-proportionality of quantity change:

$$K_{t}^{AD} = \sum_{i=1}^{N} \frac{1}{N} \left( d \ln(m_{it}) - \left( \sum_{j=1}^{N} d \ln(m_{jt}) / N \right) \right)^{2}$$  
$$K_{t}^{T} = \sum_{i=1}^{N} s_{it} \left( d \ln(m_{it}) - \left( \sum_{j=1}^{N} s_{jt} d \ln(m_{jt}) \right) \right)^{2}$$

The latter measure is a continuous time version of the Divisia quantity second moment defined by Theil (1967). We refer to these measures as the Allen-Diewert and Theil quantity variances.

### III.2 Simulation Analysis

We show that these price and quantity variances are correlated with the tracking error using simulated data. We use simulated data to avoid other sources of error that are present in empirical data. The major sources of additional error in aggregate monetary data are 1) measurement error, 2) misidentification of weakly separable blocks of monetary assets, 3) problems caused by aggregation across agents, and 4) problems caused by aggregating across sectors, such as households and firms. Our theorems only deal with error caused by inappropriate use of the summation formula, but all of these problems are likely to contribute to errors in real world data. Finally, we cannot directly observe the quantity aggregate using real data, but can (at best) approximate it using a superlative index number. We simulated a household monetary asset demand system, which allows us to develop insights more clearly and avoids the other sources of error.

The simulations were created as follows. The first step was to obtain estimates of monetary asset holdings and user cost prices for the household sector using data from the Flow of Funds Accounts. The second step was to estimate a household demand system using this household dataset. We assumed that household preferences were CES, and estimated the parameters using maximum likelihood. The third step was to simulate the demand system using the estimated parameters. We simulated price and expenditure data using a vector auto regression that was estimated from the Flow of Funds data. We provide a detailed discussion of these steps in
the appendix to this paper. The household demand system has three composite monetary goods: Checkable Deposits and Currency, Small Time and Savings Accounts, and Open Market Paper and Treasury Securities, corresponding to the asset categories available from the Flow of Funds Accounts. We simulated the system 100 times, with 133 observations for each simulation. We simulated the system over a range of implied elasticities of substitution.

In addition, we simulated demand systems that disaggregate the three composite goods into nine elementary goods: Currency, Demand Deposits, Other Checkable Deposits, Small-Denomination Time Deposits, Savings Deposits, Money Market Funds, Treasury Securities, Savings Bonds, and Open Market Paper. The disaggregated simulation more closely mirrors the disaggregated assets contained in a broad monetary aggregate such as M2 or M3. We assumed that preferences have a nested CES structure as described in the Appendix. The nested CES structure implies that Currency, Demand Deposits, and Other Checkable Deposits are weakly separable from the other goods with CES sub-utility function, Small-Denomination Time Deposits, Savings Deposits are weakly separable from the other goods with CES sub-utility function, and Money Market Funds, Treasury Securities, Savings Bonds, and Open Market Paper are weakly separable with CES sub-utility function. The sub-aggregates are composite goods, and we assume that the utility function for these 3 composite goods is also CES. If the purpose of this study were to produce valid demand system estimates we would need to test these weak separability assumptions, but we are only using them as the basis for a simulation. We note, however, that the form of the data in the Flow of Funds implies this weak separability structure.

The representation theorems suggest that the Theil and Allen-Diewert variances will be correlated with the squared tracking error in the summation aggregate. We constructed summation aggregates for each simulation, and computed the discrete time squared tracking error, \( \left( \Delta \ln(S(m_t)) - \Delta \ln(U(m_t)) \right)^2 \). We tested for the significance of correlation between the squared tracking error and \( J^T_i, J^{AD}_i, K^T_i, K^{AD}_i \). The average of the contemporaneous correlations across simulations are given in Table 1 for the three good systems, and Table 2 for the nine good systems.

We found that both the Theil and Allen-Diewert variances were highly correlated with the squared error in the three composite good systems. We simulated the three composite good systems over a range of elasticities of substitution implied by a two standard deviation confidence interval around the maximum likelihood estimate. The value \( \sigma = .17 \) is the estimated elasticity of
substitution, and the range (.11,.27) is a two standard deviation confidence interval. We also include a high value of .37 for comparison. Discussions of summation aggregates often focus on the exact perfect substitution case, but we found that the data are much closer to the other exact case – Leontief utility. The correlation between the tracking error and price and quantity variances is the same for the three good systems. This is because the price and quantity variances are perfectly co-linear for CES preferences (see Appendix A.5 for proof), although this is not generally true. Thus, for CES preferences the quantity and price variances contain the exactly the same information.

The variances are not perfectly co-linear for the nine good nested CES systems. The nine good system is simulated using \( \sigma = .17 \) and .27 and over a range of elasticities of substitution for the nested CES sub-utility functions. We considered the range of values considered for the three composite goods, but we also considered some values that are much closer to perfect substitutability. We considered a range of values between 1.0 and 2.66. We again find that there is substantial correlation between the price and quantity variances and the squared error. The correlation is statistically significant in most cases. Detailed Ljung-Box test statistics for each simulation are available from the authors upon request.

We also provide average correlations between the percentage change of the summation aggregate, \( \Delta \ln(S(m_i)) \), and the percentage change in the economic quantity aggregate, \( \Delta \ln(U(m_i)) \) in Tables 3 and 4. The correlation can be quite high for the cases in which the elasticity of substitution is close to the exact Leontief case, but can be very low for some cases. For comparison, we also compute the average correlation between the percentage change in a superlative quantity index number, \( \Delta \ln(MSI_i) \), and the percentage change in the microeconomic quantity aggregate, \( \Delta \ln(U(m_i)) \). There are many possible choices for a superlative index. We used the Tornqvist-Theil discrete time approximation to the continuous time Divisia index:

\[
\Delta \ln(MSI_i) = \sum_{t=1}^{N} \bar{s}_{it} \cdot (\ln(m_i) - \ln(m_{i-1}))
\]

where \( \bar{s}_{it} = (s_i + s_{i-1})/2 \) are the average expenditure shares. Tables 3 and 4 show that the tracking ability of the superlative index is uniform across the different cases, and is better on average than the summation aggregate.
IV. Empirical Results and Policy Implications

In this section, we investigate the implications of our results using empirical data for the United States. We cannot directly observe the monetary quantity aggregate using real data, so we must form an estimate of it using a superlative index number. The superlative index number is a second-order approximation to the underlying monetary aggregate.\(^{16}\) We used the Torqvist-Theil quantity indexes produced by Andersen, Jones, and Nesmith (1997), which are published by the Federal Reserve Bank of St. Louis. As mentioned in Section III.2, the Torqvist-Theil quantity index is defined by the equation

\[
\Delta \ln(\text{MSI}_t) = \sum_{i=1}^{N} \bar{s}_it \cdot \left( \ln(m_{it}) - \ln(m_{i,t-1}) \right) = \sum_{i=1}^{N} \bar{s}_it \cdot \Delta \ln(m_{it}) ,
\]

where \( \bar{s}_it = (s_{it} + s_{i,t-1})/2 \) are the average expenditure shares. We can approximate the tracking error of the summation aggregate in discrete time from the primal representation:

\[
\Delta \ln(\varepsilon_t) = \sum_{i=1}^{N} \left( w_{it} - \bar{s}_{it} \right) \cdot \Delta \ln(m_{it}) .
\]

We can define discrete time versions of the Theil variances using discrete changes and average expenditure shares as follows:

\[
J^T_t = \sum_{i=1}^{N} \bar{s}_it \cdot \left( \Delta \ln(\pi_{it}) - \left( \sum_{j=1}^{N} \bar{s}_jt \cdot \Delta \ln(\pi_{jt}) \right) \right)^2 ,
\]

\[
K^T_t = \sum_{i=1}^{N} \bar{s}_it \cdot \left( \Delta \ln(m_{it}) - \left( \sum_{j=1}^{N} \bar{s}_jt \cdot \Delta \ln(m_{jt}) \right) \right)^2 .
\]

The variables \( K^T_t \) and \( J^T_t \) are also published by the Federal Reserve Bank of St. Louis. We also constructed the Allen-Diewert variances using the same data. We found that for the United States data the measures are very similar except during periods when new assets are introduced.\(^{17}\) When new assets are introduced they typically have very small assets stocks. The user costs of new assets are often more volatile than existing user costs. The new assets will not contribute very much to the magnitude of the Theil variances, because their expenditure shares are very small. The new assets can contribute significantly to the magnitude of the Allen-Diewert variances, because all assets are weighted equally. Diewert (1995a) suggested share weighting the elements of the Allen-Diewert measures, which produces the Theil measures. We only report results for the Theil variances in this section.\(^{18}\)
IV.1 Empirical Results

We investigated the relationship between the squared tracking error and the Theil price and quantity variances using quarterly data for the United States. We provide results at the M1 and M2 levels of aggregation. The results at the M3 level of aggregation are similar to those at the M2 level and are available upon request. We found that the squared tracking error was highly correlated with both of the Theil variances. The main difference between the simulated results in Section III and the empirical results is that for the United States data the squared tracking error can actually be predicted using past values of the Theil variances, which was not true for our simulations. We provide the cross correlations between the squared tracking error and the variances in Table 5 for leads and lags up to ±4 quarters. Q tests for significance of correlation between the squared tracking error and lagged values of the Theil variances suggested that the variances may be able to predict the tracking error. We investigated this possibility by running Granger causality tests, which are provided in Table 6. The tests of Granger causality imply that both variances would be able to predict the squared tracking error at the M2 level of aggregation and that the price variance would be able to predict the squared tracking error at the M1 level of aggregation. These results are consistent with the cross correlations from Table 5.

IV.2 Monetary Policy Implications

The results provided in this paper have implications for the conduct and assessment of monetary policy. We focus on several instances where our results can help interpret the historical record: the case of the “missing money” from 1972-1976, and the “monetarist experiment” from 1978-1983. Table 7 provides important summary information referred to throughout this section.

The Case of the Missing Money 1972-1976

Goldfeld (1976) showed that forecasting equations for summation M1 based on data from 1952-1974 failed to produce acceptable out of sample predictions for M1. These forecasting equations consistently over predicted the value of summation M1 during the subsequent period 1974-1976, giving rise to the term “missing money”. One of the proposed solutions to the “missing money” case was that substitution had occurred into the components (such as small denomination time deposits) of a broader monetary aggregate such as M2. We can use our results to further illuminate this part of the historical record. The squared tracking error for summation M1 increased dramatically during the 1972-1976 period relative to previous levels. The price and quantity variances for M1 were also unusually high during this period, indicating a greater degree
of non-proportional price and quantity changes. If we compare the performance of summation M2 with summation M1 over this same period, we see that the story is quite different.

The data show that the squared tracking error for summation M2 was actually lower during the 1972-1976 period than during the previous period as were the price and quantity variances. We can see that one potential cause of the poor out of sample performance of summation M1 was measurement error in the summation aggregate that was caused by an unusually high degree of non-proportional price and quantity change. The tracking error of summation M1 became more negative during the 1972-1976 period, which could be responsible for at least some of the “missing money”.

The Monetarist Experiment 1978-1982

The Federal Reserve began targeting non-borrowed reserves in 1978. The purpose of the new regime was to target the summation M1 and M2 monetary quantity aggregates. It has been widely documented that interest rate volatility increased markedly during the next several years as a consequence of the change in operating procedures. The non-borrowed reserves targeting regime was abandoned by 1984. We provide summary data on the periods 1960:2-1969:4, 1970:1-1978:2, 1978:3-1983:4, 1984:1-2001:2, and 1960:2-2001:2 for both M1 and M2.19

It is clear from Table 7 that the degree of non-proportional price change is very high during the period 1978:3-1983:4. The greater interest rate volatility that has been observed during this period is co-incident with an increase in the price variance for the monetary assets. It is also clear from the tables that the squared tracking error of the summation aggregate is extremely large during the period 1978:3-1983:4, especially at the M2 level of aggregation. Additionally, the growth rate of summation M2 greatly exceeded the growth rate of the superlative index over the period 1978:3-1983:4. In other words, monetary policy was far more contractionary than would have been perceived based on the summation aggregates.20

V. Conclusion

We derived primal and dual representations of the tracking error in summation monetary aggregates. This representation connects the tracking error of summation aggregates to the time series properties of the vectors of interest rates, user costs, and asset stocks. Our results indicate that changes in the economic environment, such as monetary policy or financial innovation, which increase the degree of non-proportional price change will lead to increased measurement error in the summation aggregates. The price and quantity variances used in this paper to measure the
degree of non-proportional price and quantity change could be included in empirical models that had been previously estimated using summation aggregates as a test for measurement error. If the price and quantity variances are statistically significant it indicates that the model is affected by measurement error, and more importantly that the results of the model could be invalid.\textsuperscript{21}

Summation aggregates are commonly used in empirical analyses of events, such as changes in monetary policy, financial innovation, or regulatory changes, that have been historically associated with increased interest rate volatility, and consequently increased non-proportional price change. For example, we showed that the degree of non-proportional price change was very high in the United States during the period 1978-1983, which is partially attributable to changes in the monetary policy regime. Because, the measurement error in summation aggregates may change systematically in response to monetary policy changes, the results of any empirical study that does not account for the interaction between monetary policy and measurement error could be severely damaged. In the event of a shift in the policy regime, structural relationships that are immune to the Lucas critique may appear to breakdown if they are measured with summation monetary aggregates. The analysis presented in this paper suggests that it is preferable to use superlative monetary index numbers, because they are not subject to large time varying tracking errors.

**Appendix**

**A.1 Household Monetary Aggregates**

The Flow of Funds Accounts (FOF) provide detailed information regarding the monetary asset holdings of the household sector. The FOF breaks down the household sector’s financial asset holdings into various categories. Of these, five of the categories are included in the monetary aggregate L, and are treated as monetary assets in this paper: checkable deposits and currency (CDC); small time and savings accounts (STSA); money market fund shares (MMF); open market paper (OMP); Savings Bonds (SB); and Treasury securities (TS).\textsuperscript{22} These asset stocks must be paired with prices (user costs). This creates a problem because some of the asset categories combine several types of assets. For example, CDC contains some interest bearing checkable deposits and some non-interest bearing assets such as currency and demand deposits. The MSI database of the Federal Reserve Bank of St. Louis publishes prices for all assets in the monetary aggregates defined by the Board of Governors of the Federal Reserve System, but there is no way to break the asset quantities from the FOF up into finer categories corresponding directly to these
prices. Consequently, we form unilateral index numbers over the prices of the assets in each category based on Diewert (1995b). We apply the Jevons unilateral price index formula, defined by Diewert (1995b). This formula is based on the assumption that the components of each category (for example, checkable deposits and currency) have elasticity of substitution equal to one, which seems to be more reasonable than other available alternatives. This procedure allows us to construct a price for each of the five categories. The quantities are converted to per-capita terms. Several of the categories contain assets with high interest rates, and low prices. Consequently, MMF, TS, SB, and OMP have very small expenditure shares. We aggregated these assets into a single asset category called TCM, which has a smaller expenditure share than either CDC or STSA. The demand system estimation in the next section is based on the three composite goods.

A.2 Household Demand System Estimation

In order to focus on the key issues discussed in the paper, we needed to estimate a demand system that imposes global regularity. If regularity were violated at some price/expenditure combinations then the necessary conditions for microeconomic aggregation across goods would be violated. Consequently, we estimate a simple demand system based on the generalized CES direct utility function. The utility function has the form:

$$U(m_1, m_2, m_3) = \left( a_1(m_1 - \alpha_1)^\rho + a_2(m_2 - \alpha_2)^\rho + a_3(m_3 - \alpha_3)^\rho \right)^{1/\rho},$$

assuming that $\rho < 1$. The share equations for the CES have the form:

$$\hat{s}_i = \frac{\pi_i \alpha_i}{y} + \frac{\pi_i \left( \frac{m_i}{a_i} \right)^{1/(\rho - 1)}}{\pi_1 \left( \frac{m_1}{a_1} \right)^{1/(\rho - 1)} + \pi_2 \left( \frac{m_2}{a_2} \right)^{1/(\rho - 1)} + \pi_3 \left( \frac{m_3}{a_3} \right)^{1/(\rho - 1)} \cdot \left(1 - \frac{\pi_1 \alpha_1}{y} - \frac{\pi_2 \alpha_2}{y} - \frac{\pi_3 \alpha_3}{y}\right)}$$

$$= \frac{\pi_i \alpha_i}{y} + \frac{\psi_i \left( \frac{m_i}{a_i} \right)^{(\rho - 1)}}{\psi_1 \left( \frac{m_1}{a_1} \right)^{(\rho - 1)} + \psi_2 \left( \frac{m_2}{a_2} \right)^{(\rho - 1)} + \psi_3 \left( \frac{m_3}{a_3} \right)^{(\rho - 1)} \cdot \left(1 - \frac{\pi_1 \alpha_1}{y} - \frac{\pi_2 \alpha_2}{y} - \frac{\pi_3 \alpha_3}{y}\right)}$$

where $\psi_i = (1/a_i)^\rho$. Only two of the three parameters $a_1, a_2, a_3$ are uniquely determined, the other represents an arbitrary normalization of the utility function. We set $\psi_3 = 1$ ($a_3 = 1$) and estimate the remaining parameters. The sum of the shares is identically equal to 1, so we drop the TCM share from the estimation. We impose the constraints $\psi_1 > 0, \psi_2 > 0, \rho < .95$ by setting $\psi_1 = e^{\lambda_1} > 0, \psi_2 = e^{\lambda_2} > 0, \rho = .95 - e^{\lambda_3} < .95$. We assume that $\epsilon_i = \left( \frac{s_{it} - \hat{s}_{it}}{s_{it} - \hat{s}_{it}} \right)$ is generated by a stationary vector autoregression with i.i.d. vector Gaussian data generating process, so that
\( \varepsilon_i = R \varepsilon_{i-1} + \xi_i \quad \text{with} \quad \xi_i \sim N(0, \Sigma) \). We estimate \( \lambda_1, \lambda_2, \alpha_1, \alpha_2, \alpha_3, r_{11}, r_{12}, r_{21}, r_{22} \) using maximum likelihood estimation.\(^{28}\) We provide estimates for \( \lambda_1, \lambda_2, \lambda_3, \alpha_1, \alpha_2, \alpha_3 \) in Table A1, and estimates for \( \lambda_1, \lambda_2, \lambda_3 \) under the restrictions that \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \) in Table A2. The likelihood ratio test of the restrictions is \( 3.7632 \sim \chi^2(3) \) under the null. We would not reject the restrictions and therefore regard the restricted model as correct. We find low estimates of the implied elasticity of substitution, with fairly wide confidence intervals for \( \lambda_3 \). We construct simulations over a range of values for \( \lambda_3 \) based on two standard deviation confidence intervals around the point estimates, and therefore over a wide range of implied elasticities of substitution. The estimations are based on a sample of quarterly data from 1967:1-2000:1.\(^{29}\)

**A.3 Simulations of the Household Demand System**

We simulate the price and expenditure series and use these series to generate asset stocks (quantities) based on the Marshallian demand functions for the CES utility function. We estimated a first-order VAR over the first differenced vector of user cost prices and per-capita expenditure as logarithms.\(^{30}\) We then produced simulated data from the estimated VAR using random simulations of a vector Gaussian process. The mean and covariance matrix of the vector Gaussian process is the same as the mean and covariance matrix of the residuals from the estimated VAR.\(^{31}\) The simulated data were then converted back to levels to produce simulated price and expenditure series.\(^{32}\)

**A.4 Simulations with Sub-Aggregation Considered**

It is implicitly assumed that the components of the composite goods in the FOF are weakly separable from the other monetary assets that make up the households monetary portfolio, and the composite goods are summation sub-aggregates. It is not possible to disaggregate the data, but we wanted to simulate a demand system with more goods. We assume that CDC is composed of currency, demand deposits, and other checkable deposits, STSA is composed of small-denomination time deposits and savings deposits, and TCM is composed of MMF, TS, SB, and OMP. We estimated a 10 variable first-order VAR over the first differenced vector of prices (obtained from the Federal Reserve Bank of St. Louis) and per-capita expenditure as logarithms. We simulate the prices and expenditure in the same way described in A.3.

We assume that the utility function has a nested CES form:
\[ U(m) = V(g_1(m_1), g_2(m_2), g_3(m_3)), \]

where \( V(x, y, z) \) is the CES utility function estimated in A.2. The category sub-utility functions are assumed to be CES of the form:

\[ g_i(m_i) = \left( m_{i1}^{\rho_i} + \ldots + m_{in_i}^{\rho_i} \right)^{1/\rho_i}, \]

where \( N_i \) is the number of monetary goods in sub-aggregate \( i \). The per-capita quantities of the monetary goods are then simulated using the Marshallian demand functions based on the nested CES demand system. The parameters \( \rho_i \) \((i = 1, 2, 3)\) cannot be estimated, because we do not have disaggregated data. Instead, we simulated the system using a variety of values, as described in the text.

**A.5 Proof that the price and quantity variances are perfectly co-linear for CES preferences**

The demand functions for a 3 good CES demand system are as follows:

\[ q_i = \frac{\psi_i(\pi_i)^{\rho_i(\rho-1)}}{\psi_1(\pi_1)^{\rho_1(\rho-1)} + \psi_2(\pi_2)^{\rho_2(\rho-1)} + \psi_3(\pi_3)^{\rho_3(\rho-1)}} \cdot \frac{y}{\pi_i} \]

Thus, the log differentials have the form:

\[ d \ln(q_i) = d \ln(\phi_i) + d \ln(y) - \frac{1}{\rho - 1} d \ln(p_i) - d \ln(\sum_{i=1}^{3} \psi_j(\pi_j)^{\rho_j(\rho-1)}) \]

Let \( d \ln(\phi_i) = \sum_{i=1}^{3} \phi_i d \ln(q_i) \), where \( \sum_{i=1}^{3} \phi_i = 1 \) be a weighted sum of the log differentials. The terms of the weighted sum have the form:

\[ \phi_i \left( d \ln(q_i) - d \ln(\phi_i) \right) = \phi_i \left( \frac{1}{\rho - 1} d \ln(\pi_i) - \frac{1}{\rho - 1} \sum_{i=1}^{3} \phi_i d \ln(\pi_i) \right) \]

\[ = \frac{\phi_i}{\rho - 1} \left( d \ln(\pi_i) - \sum_{i=1}^{3} \phi_i d \ln(\pi_i) \right) \]

This establishes that the quantity and price variances are co-linear.
Endnotes

1 For clarity of exposition, we provide discussions from the standpoint of consumer aggregation theory.

2 Seminal papers in this area include Diewert (1974), Barnett (1978,1980), and Donovan (1978).

3 See Anderson, Jones, and Nesmith (1997) for details on superlative monetary index numbers for the United States. Barnett (1982) provides theory that can be used to choose the optimal level of aggregation.

4 Superlative index numbers were first defined by Diewert (1976). Superlative index numbers provide second order approximations to the underlying economic aggregates. See Barnett, Offenbacher, and Spindt (1984), Belongia (1996), and Barnett and Serletis (2000) for empirical comparisons of summation monetary aggregates versus superlative index numbers.

5 We define tracking error formally in Section II.2. The tracking error is the error in the growth rate of the summation aggregate.


7 Overviews of aggregation theory in the general case can be found in Diewert (1981), Blackorby, Primont, and Russell (1978), and Pollak (1971). In the case of monetary aggregation theory, one should consult Anderson, Jones, and Nesmith (1997) and Barnett and Serletis (2000).

8 This property is called strong decentralizability. If the full decision is strongly decentralizable, the utility function in (2) is homothetic, and the decision satisfies the condition of additive price aggregation, then multi-stage allocation theory can be used to simplify the full decision. See Theorems 5.2 and 5.8 in Blackorby, Primont, and Russell (1978, pp. 188 and 206) for full details.

9 Dutkowsky (1999) examines the impact of taxation on monetary aggregation theory and index number methods.

10 We have been assuming that U satisfies ) – iv). If i) – iii) are satisfied, but not homotheticity, then the price aggregate is the expenditure function defined on a referent utility level and the quantity aggregate is the dual distance function defined on the same referent utility level. The definitions of the exact indexes are adjusted accordingly. In the homothetic case, the value of the exact indexes is independent of the referent utility level. In the non-homothetic case, strong factor reversal will not hold in general. See Blackorby, Primont and Russell (1978) and Barnett (1987) for more discussion.


12 The factor reversal property (7) holds because we made a minor modification of the sum opportunity cost proposed by Moore, Porter, and Small. We discount both the benchmark rate and the own rate on the sum aggregate to present value, and convert the difference to a nominal aggregate.

13 This is because of the homogeneity (of degree zero) of the Hicksian demands.

14 More generally, it can be shown that there is an identity connecting the price and quantity variances with a quantity/price covariance and a variance associated with the weights. The weight variance is zero for the Allen-Diewert measures (because they do not change) and is a share variance for the Theil measure. The identity is proven in Theil (1967) for the share weighted measures and can be extended easily to the Allen-Diewert measures.

15 Barnett (1980) found an elasticity of substitution of 2.66 for passbook accounts at different institution types. This high degree of substitutability is taken as an upper bound for different types of assets.

16 The summation aggregate is exact if the assets are perfect substitutes or if the utility function is Leontief. Superlative index numbers were defined by Diewert (1976) as being exact for a flexible functional form, and thus provide a second-order approximation to a general aggregator function.

17 The introduction of new assets into the monetary aggregates is discussed in Andersen, Jones, and Nesmith (1997, pp79). Of particular importance is the introduction of retail money funds and OCD accounts in the early 1970’s and the introduction of MMDAs in the early 1980’s.

18 The Allen-Diewert variances performed about as well as the Theil variances in our simulation, because all assets had reasonably large expenditure shares, and the components of the aggregates did not change. There are several possible ways of resolving the problem of new assets. One way is to share weight the component price changes, producing the Theil variances. Alternatively, we could have sub-aggregated the new assets with a close substitute asset and consider the price change of the aggregate good. In practice, this is very similar to share weighting, because the sub-aggregate would be computed as a superlative index such as Tornqvist-Theil. The other possible solution would be to simply omit the new asset, until it obtained a reasonable expenditure share. We found that the Allen-Diewert variances were very highly correlated with the Theil variances except during the periods where new assets were being introduced into the aggregate.

19 We divided the 1960’s from the 1970’s because of issues of user cost data quality and to highlight the rather dramatic change in the growth rates of the monetary quantity aggregates immediately before and after the change in operating procedures.

20 Barnett (1987) has commented on the latter point at length.

21 This test is similar to the dispersion dependency test in Barnett and Serletis (1990).

22 L was defined as M3 plus stocks of Short-term Treasury Bills, Commercial Paper, Savings Bonds, and Banker’s Acceptances.

23 See Anderson, Jones, and Nesmith (1997, pp. 76-77) for details.

24 The Jevons formula also assumes that the expenditure shares within the sub-aggregate are constant. The main alternative is what Diewert called a Dutot index. The Dutot formula assumes that the elasticity of substitution is zero and the quantities of the assets
within the sub-aggregate are all equal. Anderson, Jones, and Nesmith (1997) suggest a third possibility based on the assumption that the components of the sub-aggregate are perfect substitutes.

25 The CDC user cost price is a unilateral index constructed from user cost prices for currency, demand deposits, other checkable deposits, and super NOW accounts. The STSA user cost price is a unilateral index constructed from user cost prices for savings accounts, money market deposit accounts, small time deposits, and large time deposits. The user cost price for MMF is the user cost price for broker dealer money market mutual fund shares. The user cost price for OMP is a unilateral price index constructed from the user cost prices for Banker’s acceptances, and commercial paper. The user cost price for TS is a unilateral price index constructed from user cost prices on savings bonds and short-term treasury securities.

26 The average expenditure share for CDC is 0.24, the average expenditure share for STSA is 0.65, and the average expenditure share for TCM is 0.1.

27 A plausible alternative would have been to estimate a demand system with a large regularity region and analyze this system over points where regularity is satisfied. A possible functional form with this property is the min-flex Laurent model. See Barnet and Lee (1995).

28 We actually estimate the system using two different procedures. The first ignores the distribution of the initial observation, which has been called a generalized first difference method; see Pollack and Wales (1992). The second assumes that the VAR is stationary and accounts for the distribution of the initial observation. When this method is used we assume that the matrix \( R \) is diagonal, allowing only one free parameter. In practice the estimates are quite similar for most parameters. We report estimates based on the first method. We obtained estimates using a conjugate gradient algorithm. The standard errors are produced from an estimate of the inverse of the information matrix. Our estimate is based on summations (across all data points) of the outer product of the gradient of the density function with itself.

29 We omitted the first few years of data due to some anomalous looking data points.

30 The estimated levels VAR produced unit roots.

31 The initial values for the simulated VAR are vector Gaussian random variables with mean and covariance matrix computed from the stationary distributions of the estimated VAR. The covariance matrix of the stationary distribution is computed using a doubling algorithm. The simulated input process is computed from \( \xi_t \) an i.i.d. mean zero vector Gaussian process with identity covariance matrix. The process is transformed into \( \eta_t = \mu + C \xi_t \), where \( \mu \) is the mean and \( CC^T \) is the covariance matrix for the residuals of the estimated VAR. We used the Cholesky factorization to produce \( C \).

32 The levels are produced by choosing an initial value for the logarithm of the levels and then accumulating the values from the simulated VAR (which was estimated to mimic the log first differences of the actual data). The resulting data is converted from logarithms into levels. The initial value is basically arbitrary, but we chose the value of the actual data several quarters earlier.
References
Dutkowsky, Donald H. “Taxation and Monetary Aggregation” Journal of Money, Credit, and Banking v31, n4 (November 1999): 811-17
Table 1: Average Contemporaneous Cross Correlations

Tracking Error Versus Price and Quantity Variances from Simulation

3 Composite Good Simulations, CES Utility

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<th>$J_i^T, \Delta (\ln(e_i))^2$</th>
<th>$J_i^{AD}, \Delta (\ln(e_i))^2$</th>
<th>$K_i^T, \Delta (\ln(e_i))^2$</th>
<th>$K_i^{AD}, \Delta (\ln(e_i))^2$</th>
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<td>.73191</td>
<td>.69179</td>
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<tr>
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<td>.69270</td>
<td>.73056</td>
<td>.69270</td>
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<tr>
<td>$\sigma = .37$</td>
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<td>.69668</td>
<td>.73060</td>
<td>.69668</td>
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</tbody>
</table>

$\sigma = 1/(1 - \rho)$ is the elasticity of substitution between the 3 composite goods, see Appendix A.2
Table 2: Average Contemporaneous Cross Correlations from Simulation
Tracking Error Versus Price and Quantity Variances
9 Good Simulations, Nested CES Utility

<table>
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<tr>
<th>( \sigma = .17 )</th>
<th>( J_t^T, \Delta (\ln(\epsilon_t))^2 )</th>
<th>( J_t^{AD}, \Delta (\ln(\epsilon_t))^2 )</th>
<th>( K_t^T, \Delta (\ln(\epsilon_t))^2 )</th>
<th>( K_t^{AD}, \Delta (\ln(\epsilon_t))^2 )</th>
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<tbody>
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<td>.43576</td>
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<td>.51973</td>
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<td>.33017</td>
<td>.43576</td>
<td>.39135</td>
<td>.51973</td>
</tr>
<tr>
<td>( \sigma_1 = \sigma_2 = \sigma_3 = .37 )</td>
<td>.29706</td>
<td>.49007</td>
<td>.31675</td>
<td>.53078</td>
</tr>
<tr>
<td>( \sigma_1 = \sigma_2 = \sigma_3 = 2.66 )</td>
<td>.49237</td>
<td>.52375</td>
<td>.39055</td>
<td>.48898</td>
</tr>
<tr>
<td>( \sigma_1 = \sigma_2 = \sigma_3 = 2. )</td>
<td>.44609</td>
<td>.53431</td>
<td>.36444</td>
<td>.53151</td>
</tr>
<tr>
<td>( \sigma_1 = \sigma_2 = \sigma_3 = 1.5 )</td>
<td>.37899</td>
<td>.47823</td>
<td>.34979</td>
<td>.46352</td>
</tr>
<tr>
<td>( \sigma_1 = \sigma_2 = \sigma_3 = 1.1 )</td>
<td>.55778</td>
<td>.26360</td>
<td>.55587</td>
<td>.15397</td>
</tr>
<tr>
<td>( \sigma_1 = \sigma_2 = \sigma_3 = .11 )</td>
<td>.38312</td>
<td>.38908</td>
<td>.41515</td>
<td>.57670</td>
</tr>
<tr>
<td>( \sigma_1 = \sigma_2 = \sigma_3 = .17 )</td>
<td>.37695</td>
<td>.45616</td>
<td>.37882</td>
<td>.52734</td>
</tr>
<tr>
<td>( \sigma_1 = \sigma_2 = \sigma_3 = .27 )</td>
<td>.34974</td>
<td>.50248</td>
<td>.34974</td>
<td>.50248</td>
</tr>
<tr>
<td>( \sigma_1 = \sigma_2 = \sigma_3 = .37 )</td>
<td>.31552</td>
<td>.50521</td>
<td>.32659</td>
<td>.51281</td>
</tr>
<tr>
<td>( \sigma_1 = \sigma_2 = \sigma_3 = 2.66 )</td>
<td>.48658</td>
<td>.55837</td>
<td>.33447</td>
<td>.54358</td>
</tr>
<tr>
<td>( \sigma_1 = \sigma_2 = \sigma_3 = 2. )</td>
<td>.43871</td>
<td>.54457</td>
<td>.32960</td>
<td>.54726</td>
</tr>
<tr>
<td>( \sigma_1 = \sigma_2 = \sigma_3 = 1.5 )</td>
<td>.36385</td>
<td>.48246</td>
<td>.32427</td>
<td>.46427</td>
</tr>
<tr>
<td>( \sigma_1 = \sigma_2 = \sigma_3 = 1.1 )</td>
<td>.54843</td>
<td>.27374</td>
<td>.54610</td>
<td>.17157</td>
</tr>
</tbody>
</table>

\( \sigma = 1/(1 - \rho) \) is the elasticity of substitution between the 3 composite goods, see Appendix A.2. \( \sigma_i = 1/(1 - \rho_i) \) is the elasticity of substitution between the components of the \( i \)th weakly separable block, see Appendix A.4.
Table 3: Average Contemporaneous Cross Correlations from Simulation

Summation Aggregates and Superlative Index Numbers Versus Quantity Aggregate

3 Composite Good Simulations, CES Utility

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\Delta \ln(S(m_j)), \Delta \ln(U(m_j))$</th>
<th>$\Delta \ln(MSI_j), \Delta \ln(U(m_j))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.11</td>
<td>.99184</td>
<td>.99735</td>
</tr>
<tr>
<td>.17</td>
<td>.97631</td>
<td>.99275</td>
</tr>
<tr>
<td>.27</td>
<td>.92626</td>
<td>.97922</td>
</tr>
<tr>
<td>.37</td>
<td>.83680</td>
<td>.95587</td>
</tr>
</tbody>
</table>

$\sigma = 1/(1 - \rho)$ is the elasticity of substitution between the 3 composite goods, see Appendix A.2
Table 4: Average Contemporaneous Cross Correlations from Simulation
Summation Aggregates and Superlative Index Numbers Versus Quantity Aggregate
9 Goods, Nested CES Utility

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\Delta \ln(S(m_i)), \Delta \ln(U(m_i))$</th>
<th>$\Delta \ln(MSI_i), \Delta \ln(U(m_i))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = .17$</td>
<td>.99417</td>
<td>.99880</td>
</tr>
<tr>
<td>$\sigma_1 = \sigma_2 = \sigma_3 = .11$</td>
<td>.98969</td>
<td>.99880</td>
</tr>
<tr>
<td>$\sigma = .17$</td>
<td>.96704</td>
<td>.99881</td>
</tr>
<tr>
<td>$\sigma_1 = \sigma_2 = \sigma_3 = .17$</td>
<td>.89012</td>
<td>.99883</td>
</tr>
<tr>
<td>$\sigma = .17$</td>
<td>.80775</td>
<td>.99624</td>
</tr>
<tr>
<td>$\sigma_1 = \sigma_2 = \sigma_3 = .37$</td>
<td>.74354</td>
<td>.99687</td>
</tr>
<tr>
<td>$\sigma = .17$</td>
<td>.73717</td>
<td>.99775</td>
</tr>
<tr>
<td>$\sigma_1 = \sigma_2 = \sigma_3 = 2$</td>
<td>.73717</td>
<td>.99996</td>
</tr>
<tr>
<td>$\sigma = .17$</td>
<td>.92486</td>
<td>.99996</td>
</tr>
<tr>
<td>$\sigma_1 = \sigma_2 = \sigma_3 = 1.1$</td>
<td>.98582</td>
<td>.99664</td>
</tr>
<tr>
<td>$\sigma = .27$</td>
<td>.98067</td>
<td>.99664</td>
</tr>
<tr>
<td>$\sigma_1 = \sigma_2 = \sigma_3 = .11$</td>
<td>.95562</td>
<td>.99663</td>
</tr>
<tr>
<td>$\sigma = .27$</td>
<td>.95562</td>
<td>.99663</td>
</tr>
<tr>
<td>$\sigma_1 = \sigma_2 = \sigma_3 = .27$</td>
<td>.87197</td>
<td>.99664</td>
</tr>
<tr>
<td>$\sigma = .27$</td>
<td>.69677</td>
<td>.99013</td>
</tr>
<tr>
<td>$\sigma_1 = \sigma_2 = \sigma_3 = 2.66$</td>
<td>.62353</td>
<td>.99180</td>
</tr>
<tr>
<td>$\sigma = .27$</td>
<td>.62727</td>
<td>.99414</td>
</tr>
<tr>
<td>$\sigma_1 = \sigma_2 = \sigma_3 = 1.5$</td>
<td>.92006</td>
<td>.99980</td>
</tr>
<tr>
<td>$\sigma = .27$</td>
<td>.92006</td>
<td>.99980</td>
</tr>
</tbody>
</table>
| $\sigma_1 = \sigma_2 = \sigma_3 = 1.1$ | $\sigma = 1/(1-\rho)$ is the elasticity of substitution between the 3 composite goods, see Appendix A.2
| $\sigma_i = 1/(1-\rho_i)$ is the elasticity of substitution between the components of the $i$th weakly separable block, see Appendix A.4 |
Table 5: Cross Correlations - Squared Tracking Error Versus Price and Quantity Variances
Quarterly U.S. Data, 1960-2001

<table>
<thead>
<tr>
<th>M1 Level</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Q(1,4)</th>
<th>Q(0,4)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((\Delta \ln(S) - \Delta \ln(MSI))^2 ), (J^T)</td>
<td>.02</td>
<td>.07</td>
<td>.14</td>
<td>.26</td>
<td>.55</td>
<td>.18</td>
<td>.16</td>
<td>.32</td>
<td>.10</td>
<td>28.5727 (.0000)</td>
<td>78.7043 (.0000)</td>
</tr>
<tr>
<td>((\Delta \ln(S) - \Delta \ln(MSI))^2 ), (K^T)</td>
<td>.04</td>
<td>.13</td>
<td>.02</td>
<td>.14</td>
<td>.73</td>
<td>.10</td>
<td>.04</td>
<td>.00</td>
<td>.03</td>
<td>2.0948 (.7183)</td>
<td>89.7488 (.0000)</td>
</tr>
</tbody>
</table>

M2 Level

<table>
<thead>
<tr>
<th>M2 Level</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Q(1,4)</th>
<th>Q(0,4)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((\Delta \ln(S) - \Delta \ln(MSI))^2 ), (J^T)</td>
<td>.14</td>
<td>.15</td>
<td>.20</td>
<td>.19</td>
<td>.26</td>
<td>.34</td>
<td>.38</td>
<td>.51</td>
<td>.41</td>
<td>116.3419 (.0000)</td>
<td>127.2041 (.0000)</td>
</tr>
<tr>
<td>((\Delta \ln(S) - \Delta \ln(MSI))^2 ), (K^T)</td>
<td>.07</td>
<td>.04</td>
<td>.05</td>
<td>.06</td>
<td>.62</td>
<td>.37</td>
<td>.33</td>
<td>.26</td>
<td>.06</td>
<td>54.3133 (.0000)</td>
<td>117.8803 (.0000)</td>
</tr>
</tbody>
</table>

The Cross correlation is given by the formula 
\[
\rho_{xy}(k) = \frac{\sum (x_t - \bar{x})(y_{t+k} - \bar{y})}{\sqrt{\sum (x_t - \bar{x})^2 \sum (y_{t+k} - \bar{y})^2}},
\]
where \(x\) is squared tracking error and \(y\) is either \(J^T\) or \(K^T\) and \(k\) runs from -4 to 4.

Q(1,4) is a test of significance for the significance of lags of either \(J^T\) or \(K^T\), Q(0,4) is a test of significance of lags and current values of either \(J^T\) or \(K^T\).

Table 6: Granger Causality Tests
Quarterly U.S. Data, 1960-2001

<table>
<thead>
<tr>
<th>Excluded Variable</th>
<th>Test Statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1 Level</td>
<td></td>
</tr>
<tr>
<td>(K^T)</td>
<td>1.98014 (.1005)</td>
</tr>
<tr>
<td>(J^T)</td>
<td>9.65773 (.0000)</td>
</tr>
<tr>
<td>M2 Level</td>
<td></td>
</tr>
<tr>
<td>(K^T)</td>
<td>29.29971 (.0000)</td>
</tr>
<tr>
<td>(J^T)</td>
<td>14.15134 (.0000)</td>
</tr>
</tbody>
</table>

The test is for significance of lags 1-4 of either \(K^T\) or \(J^T\) in a regression of 
\((\Delta \ln(S) - \Delta \ln(MSI))^2\) against 4 lags of itself and either \(K^T\) or \(J^T\)
Table 7: Sample Means
Quarterly U.S. Data, Selected Periods

<table>
<thead>
<tr>
<th></th>
<th>Δ ln(SUM)</th>
<th>Δ ln(MSI)</th>
<th>(Δ ln(S) – Δ ln(MSI))^2</th>
<th>J^T</th>
<th>K^T</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>M1 Level</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1972:1-1976:4</td>
<td>5.49</td>
<td>5.88</td>
<td>.182</td>
<td>4.31</td>
<td>.65</td>
</tr>
<tr>
<td>1960:2-1969:4</td>
<td>3.84</td>
<td>3.95</td>
<td>.08</td>
<td>1.40</td>
<td>.11</td>
</tr>
<tr>
<td>1978:3-1983:4</td>
<td>7.57</td>
<td>7.81</td>
<td>1.93</td>
<td>17.01</td>
<td>17.11</td>
</tr>
<tr>
<td>1984:1-2001:2</td>
<td>4.40</td>
<td>4.64</td>
<td>1.09</td>
<td>1.05</td>
<td>2.90</td>
</tr>
<tr>
<td>1960:2-2001:2</td>
<td>5.04</td>
<td>5.25</td>
<td>.78</td>
<td>3.67</td>
<td>3.67</td>
</tr>
<tr>
<td><strong>M2 Level</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1960:2-1969:4</td>
<td>6.91</td>
<td>5.67</td>
<td>1.93</td>
<td>15.53</td>
<td>5.19</td>
</tr>
<tr>
<td>1978:3-1983:4</td>
<td>8.64</td>
<td>5.91</td>
<td>21.13</td>
<td>54.4</td>
<td>60.55</td>
</tr>
</tbody>
</table>

Δ ln(SUM) and Δ ln(MSI) are multiplied by 400, to reflect annualized percentage growth rates. J^T and K^T have been rescaled to have the same mean, but are comparable between M1 and M2.
Table A1

<table>
<thead>
<tr>
<th>Estimated Parameters</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>$\lambda_2$</td>
<td>$\lambda_3$</td>
<td>$\alpha_1$</td>
<td>$\alpha_2$</td>
</tr>
<tr>
<td>0.0703522 (0.73289)</td>
<td>1.2148347 (0.436936)</td>
<td>1.9878862 (2.556691)</td>
<td>-0.02461589 (7.648761)</td>
<td>0.032284062 (26.14348)</td>
</tr>
<tr>
<td>$r_{11}$</td>
<td>$r_{12}$</td>
<td>$r_{21}$</td>
<td>$r_{22}$</td>
<td></td>
</tr>
<tr>
<td>1.019664 (0.02325)</td>
<td>0.0114548 (0.024601)</td>
<td>-0.0440583 (0.025448)</td>
<td>0.9629498 (0.02567)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Transformed Demand Parameters</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi_1$</td>
<td>$\psi_2$</td>
<td>$\psi_3$</td>
<td>$\rho$</td>
<td>$\frac{1}{1-\rho}$</td>
</tr>
<tr>
<td>1.072886</td>
<td>3.369737</td>
<td>1.0</td>
<td>-5.721777</td>
<td>0.14877</td>
</tr>
</tbody>
</table>

Table A2

<table>
<thead>
<tr>
<th>Estimated Parameters</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>$\lambda_2$</td>
<td>$\lambda_3$</td>
<td></td>
</tr>
<tr>
<td>0.157369 (.121636)</td>
<td>1.2656098 (.130171)</td>
<td>1.77498 (.199757)</td>
<td></td>
</tr>
<tr>
<td>$r_{11}$</td>
<td>$r_{12}$</td>
<td>$r_{21}$</td>
<td>$r_{22}$</td>
</tr>
<tr>
<td>1.017152 (.020007)</td>
<td>0.0122954 (.022817)</td>
<td>-0.0383841 (.021638)</td>
<td>0.9631413 (.0235413)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Transformed Demand Parameters</th>
<th></th>
<th></th>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>$\psi_1$</td>
<td>$\psi_2$</td>
<td>$\psi_3$</td>
<td>$\rho$</td>
<td>$\frac{1}{1-\rho}$</td>
</tr>
<tr>
<td>1.170427</td>
<td>3.545254</td>
<td>1.0</td>
<td>-4.950181</td>
<td>0.1680621</td>
</tr>
</tbody>
</table>