An algorithm for sequential solutions of dynamic CGE models with perfect foresight over an infinite numbers of periods

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Abstract:

We describe a simple algorithm that permits the sequential (period-by-period) solution of large-scale dynamic CGE models with agents who have perfect foresight over an infinite number of periods. The algorithm requires neither any assumptions about behavior in a “final” period nor that the base case economy be currently in steady state. We briefly illustrate the algorithm with an analysis of substituting a flat tax for all income taxes in the United States.

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1. Introduction

Several recent papers have presented algorithms for the development of forward looking dynamic computable general equilibrium (CGE) models that do not require the assumption that the economy to be modeled is currently in a steady state. However, none of these algorithms is applicable to models in which agents have perfect foresight over an infinite number of periods. This paper fills the gap and describes a simple algorithm that permits the sequential (period-by-period) solution of large-scale dynamic perfect foresight CGE models for economies that are currently not in steady state.

Among the existing algorithms, Wendner’s (1999) algorithm is the only one to permit perfect foresight over multiple periods. His algorithm a solution for overlapping generations models where all agents are assumed to die after some time. Because his algorithm requires that all periods be solved simultaneously, it is not applicable to large-scale dynamic models that need to be solved sequentially. In addition, most CGE models describe the behavior of agents who live forever and who therefore ought to have rational expectations over their entire infinite lifetime. The algorithms by Yang (1999) and Dixon et al. (2005) fit into this general framework and allow sequential solutions, but they only permit perfect foresight over the immediately following period. To the extent that these algorithms ignore information about prices in other future periods that may follow a systematic pattern, they do not accommodate true rational expectations.

This paper is organized as follows. Section 2 describes the general formulation of dynamic perfect foresight models and uses it to illustrate the essential components of the three above-mentioned algorithms. Section 3 describes our algorithm and illustrates it with an application. Section 4 concludes.
2. The Dynamic Framework and Existing Algorithms

2.1. Perfect foresight over a finite number of periods

Consider a single representative agent with perfect foresight over $T+1$ periods with utility function

$$U = \frac{1}{1-\sigma} \left( \sum_{t=0}^{T} \left( \frac{1}{1+\rho} \right)^t C_t^{1-\sigma} + \left( \frac{1}{1+\rho} \right)^T \rho W_T^{1-\sigma} \right), \quad (1)$$

where $C_t$ is his consumption in period $t$, $W_T$ is his wealth in period $T$ (for example, his bequests), $\alpha$ measures his propensity to have wealth in period $T$ (for example, the strength of the bequest motive), $\rho$ is his pure rate of time preference, and $\sigma$ is the inverse of his intertemporal elasticity of substitution. The agent maximizes equation (1) subject to his intertemporal budget constraint

$$\sum_{t=0}^{T} p_t C_t + p_T W_T = Y, \quad (2)$$

where $p_t$ is the present value of the price index of consumption in period $t$, and $Y$ is the present value of the agent’s income over all $T+1$ periods. The first order conditions for optimal consumption in periods 0 and $t$ imply that the ratios of the marginal utility of consumption to the consumption price index must be equal for both years, or

$$\frac{\partial U / \partial C_0}{p_0} = \frac{\partial U / \partial C_t}{p_t}, \quad (3)$$

so that

$$C_t = \left( \frac{p_0}{p_t} \left( \frac{1}{1+\rho} \right)^t \right)^{1/\sigma} C_0, \quad (4a)$$
The present value of lifetime income, $Y$, is a function of factor prices in all periods. A solution of the CGE model consists of the equilibrium values of these factor prices as well as the consumption price indexes $p_t$, so that all markets clear simultaneously in all periods. Application of (4c) therefore requires that these prices be known to determine $c_0$, which makes it necessary to solve all $T + 1$ periods simultaneously.

Auerbach and Kotlikoff (1987) were the first to apply this framework in the context of an overlapping generations CGE model. They assumed that $T = 54$ for each generation and that the economy would reach a steady state in which all endogenous variables would grow at a constant rate after 250 years. They calibrated the model under the assumption that the economy is currently in such a steady state.

While this assumption is difficult to justify for most economies, it is commonly made for analytical convenience. Dynamic CGE models are generally calibrated with data from a single base year, and the assumption that the economy is currently in steady state implies that consumption grows at a constant rate from the observed value of consumption in year 0, $C_0$. This makes it straightforward to calibrate the model so that equation (4c) replicates the observed level of consumption in year 0 for any (exogenously chosen) values of $\rho$ and $\sigma$. The assumption that the economy is not in steady state leads to the difficulty that the solution for optimal consumption in year 0 that is determined
from equation (4c) may differ from the observed level of consumption. A straightforward solution proposed and implemented by Wendner (1999) is to calibrate the values of exogenous parameters like $\rho$ and $\sigma$ so that the solution to (4c) yields the observed level of consumption in year 0.$^{1}$

2.2. Perfect foresight over an infinite number of periods

The assumption of finitely lived agents is meaningful in the context of an overlapping generations model, but the majority of CGE models are built on the assumption of infinitely lived agents. Such models do not need to account for wealth in the final period so that utility function (1) reduces to the first term that describes the utility obtained from consumption in each period. Because it is not possible to calculate the present value of income over an infinite number of periods with varying price indexes, equation (4c) is no longer applicable if $T = \infty$, and a different strategy is necessary to solve the model.

One possibility is to break the infinite time horizon into two timeframes: the first covers year 0 until year $T$ and the second year $T + 1$ until infinity. If $T$ is chosen so that the economy is likely to have reached its steady state by that year, then the first $T + 1$ years describe the convergence dynamics that are generally of interest. It is therefore in

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$^{1}$ While this is very simple in principle, Wendner’s algorithm is fairly complicated because he uses the standard CGE solution algorithm GEMPACK to adjust the exogenous parameters. To be able to use this algorithm, he introduces slack variables into the intertemporal equations (4a) and (4b) whose values he sets so that the solutions for all years are identical to the solution for year 0, which yields the consistent initial dynamic solution required by GEMPACK. For any exogenously chosen values of $\rho$ and $\sigma$, consumption and savings in year 0 are likely to differ from the observed values. The next step is to set consumption (and thereby savings) in year 0 equal to the observed value, turn it into an exogenous variable, and make $\rho$ endogenous in its place. The final step is to “shock” the initial solution by setting all slack variables equal to zero and to use GEMPACK to find the new dynamic solution over $\rho$ and over all endogenous variables other than consumption in year 0.
principle possible to approximate the infinite time horizon by analyzing a finite number of years and to use Wendner’s (1999) solution method. However, his strategy requires that all \( T + 1 \) periods be solved simultaneously, which is often prohibitively time-consuming for large models.

Rather than solving the model until the economy has reached its steady state, Lau et al. (2002) argue that it may be sufficient to solve the model until it has reached some stability in growth rates, for example, until the growth of investment equals the growth of output, even if these growth rates have not converged to their steady state values. They provide evidence that for their data, the infinite horizon solution can be approximated sufficiently closely by setting \( T \) as low as 15 years. This is often sufficient to solve for equilibrium in all periods simultaneously. The main difficulty is that it is not obvious in general how many periods provide a sufficiently close approximation, and choosing \( T \) too small will have a non-negligible impact on the results.

If \( T \) is infinity (or very large), then the model needs to be solved sequentially, one period at a time. Early multi-period models consisted of a string of independently solved single-period models in which each year’s saving was independent of the actual rates of return in the following periods. Such models with myopic expectations do not accommodate forward-looking behavior and are therefore incompatible with the perfect foresight framework of equations (1) and (2). Economists have proposed two very different mechanisms that permit sequential solution of models with forward-looking agents. Both algorithms accommodate the assumption that the base case economy is not in steady state.
Yang’s (1999) mechanism incorporates forward-looking expectations by combining a series of single-period models with a separate optimal growth model. The single-period models determine the within-period equilibria, while the sole purpose of the optimal growth model is to determine the optimal aggregate consumption and savings decisions over time. The single-period models are solved sequentially and can therefore be fairly large, while the optimal growth model describes only aggregate variables and is therefore comparatively small and easy to solve.

Both models are calibrated to reflect the observed data in year 0, and the optimal growth model is solved to determine the equilibrium aggregate growth path of the model economy. The growth model’s equilibrium aggregate saving in year 1 is then used as a constraint under which the single-period model is solved for year 1. If the single-period model and the optimal growth model yield identical values for equilibrium aggregate output, then the algorithm is started all over again to determine the equilibrium in year 2, using year 1 as the new starting point. Otherwise, the factor of technological progress in the optimal growth model is adjusted and the procedure is repeated until the two models yield the same aggregate output for year 1. Because the model is calibrated and solved sequentially, it does not require that the base case economy be in steady state.

Yang’s algorithm describes forward-looking behavior but does not accommodate perfect foresight over more than one year. To reach convergence between the two models in any year, it is generally necessary to adjust the technological progress factor in the optimal growth model for that year. But the optimal growth models that were used to determine the aggregate equilibria for all earlier years obviously did not incorporate the later adjustment. This feature of the algorithm is very useful for applications that
describe unforeseeable shocks in future years that should not influence the agent’s behavior in the preceding years.\(^2\) However, in applications that do not incorporate unforeseeable shocks, it is somewhat unsatisfying that, if technological progress changes each period in a possibly systematic fashion, agents do not incorporate these predictable changes in future years into their expectations. In such applications, the algorithm therefore does not accommodate rational expectations.

Dixon et al. (2005) developed a sequential solution algorithm that in principle accommodates perfect foresight over a limited number of years. Their mechanism requires that the number of periods be limited to \(T < \infty\), and therefore only approximates a model in which agents have perfect foresight over an infinite number of periods. The first step of the algorithm is to solve the \(T + 1\) single-period models in which each period’s equilibrium saving is derived under myopic expectations about the expected rates of return in all following periods until year \(T\). Once all periods have been solved, each period’s actual rate of return is known. The sequence of single-period models is then solved again, but each period’s myopic expectations about future rates of return are adjusted to reflect the observed actual rates of return. The procedure is repeated until equilibrium saving in each period is based on expected rates of return that are identical to the actual rate of return. That is, the algorithm effectively solves for an equilibrium path where myopic expectations are identical to perfect foresight until year \(T\).

Although it overcomes the most crucial restriction of models with backward looking expectations, Dixon et al.’s (2005) algorithm still has three drawbacks. First, if saving in each period depends on the expected rates of return until period \(T\), then the

\(^2\) See, for example, Blejer et al. (2002).
number of years with perfect foresight decreases the closer one is to year $T$. This assumption is intuitive for overlapping generations models where agents are assumed to die in year $T$, but it is harder to justify in models where agents live forever and that are solved for a finite number of periods solely for numerical tractability. To avoid an endogenous change in the formation of expectations of infinitely lived agents, Dixon et al. (2005) restrict their description and application of the algorithm to the limiting case in which agents have perfect foresight only over the immediately following year.\(^3\) Again, this assumption is useful for the analysis of unforeseeable policy changes but somewhat unsatisfying in general.

Second, the model assumes that agents live forever but is solved for a finite number of periods. This makes it necessary to specify the agents’ behavior in period $T$ (the “closing condition”). For example, equation (1) introduces the propensity to have wealth in the final period, $\alpha$, to prevent the agent from consuming all assets in the final period. The choice of $\alpha$ is likely to affect the entire consumption path. This is intuitive in models with finitely lived agents who want to leave bequests. But if this parameter is introduced as an artificial closing condition into models with agents who live forever, then an assumption that is made solely for technical reasons may have a noticeable impact on the equilibrium solution.

A final drawback comes from the fact that the model is solved for only $T + 1$ periods, without any consideration of what might happen in later periods. Yang’s (1999) algorithm ensures convergence to a steady state because it requires the modeler to repeatedly solve an optimal growth model that describes the intertemporal changes in

\(^3\) It is therefore not clear whether and how quickly their algorithm will converge to equilibrium if each year’s saving were to depend on the rates of return in all future periods until year $T$. 
aggregate variables. His model leads to aggregate saving in the first year that is compatible with ultimate convergence to the steady state. In contrast, Dixon et al.’s (2005) algorithm ignores the possibility that saving will converge to either plus or minus infinity after period $T$. Such convergence to the extremes is the usual fate of models that assume exogenously chosen values for $\rho$ and $\sigma$ in conjunction with an observed (and therefore “optimal”) saving rate in year 0. Saving rates that converge to either minus or plus infinity are not compatible with rational expectations, so that this algorithm describes perfect foresight over one period but does not accommodate true rational expectations.

Because most dynamic models assume that agents have rational expectations, it is useful to have a sequential solution algorithm that accommodates true perfect foresight over an infinite number of periods without arbitrary closing conditions. We describe such an algorithm in the next section.

3. A sequential algorithm for models with agents who have perfect foresight over an infinite number of years

Our algorithm is based on the same insight as Wendner’s (1999) algorithm: if one assumes that the observed values of consumption and saving in year 0 are part of the optimal intertemporal solution, then arbitrary values of $\rho$ and $\sigma$ will cause the growth rate of savings to converge to either plus or minus infinity. It is therefore necessary to find the value of either $\rho$ or $\sigma$ which ensures that the growth rate of savings converges to a finite constant. However, while Wendner’s algorithm requires that agents have perfect

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4 In their application, they set $T = 15$ and do not investigate the further behavior of the savings path.
foresight over only a finite number of years so that all periods can be solved simultaneously, our algorithm accommodates perfect foresight over an infinite number of years and permits sequential solution.

Our algorithm is straightforward. Intertemporal optimization requires that equation (3) hold in every period. If the ratio of the marginal utility of consumption to the consumption price index in year 0 is known, then one can determine optimal consumption in year $t$ by ensuring that the ratio of marginal utility to price in year $t$ has the same value. It remains to choose a value of, say, $\rho$ that yields a ratio of marginal utility to price in year 0 such that the growth rate of savings in future years converges to a constant. Because each year is solved separately in sequence, the algorithm permits the solution of large models with many unknowns per year. Because all years are connected through the intertemporal optimality condition (3) that is derived from utility function (1) with $T = \infty$, the consumption and saving decisions in each year reflect perfect foresight for an infinite number of future periods, even if the model is solved only for a finite number of years.

The model needs to be solved for sufficiently many periods to ensure that it either has converged or is on a convergence path, but it can always be solved for another period beyond period $T$ without any change in period $T$’s optimal solution. Unlike models that are solved for a fixed number of periods, our algorithm therefore does not require any assumption about behavior in the “final period.” By eliminating the need for such closing conditions that enter directly into the utility function, our algorithm is less likely to be affected by assumptions made solely for the purpose of numerical tractability.
An essential component of the algorithm is the endogenous calibration of $\rho$, and the algorithm therefore needs to be applied slightly differently to the model calibration and to the calculation of the counterfactual solution. We next describe how the algorithm works in these two cases.

### 3.1. Algorithm to Calibrate the Dynamic Model

**Step 1:** Choose the number of years ($T$) for which to solve the model.

**Step 2:** The observed data that describe the economy in year 0 yield the equilibrium value $C_0$. Because of Walras’ law, the consumption price index in year 0 can be set to an arbitrary value (usually 1). Once values of $\rho$ and $\sigma$ are chosen (for example, from the literature on their estimation), one can determine the ratio of marginal utility to price in year 0.

**Step 3:** Use equation (4a) to determine equilibrium consumption in year $t$, $t = 1, \ldots, T$ (year $t$’s consumption price index is part of year $t$’s equilibrium solution).

**Step 4:** Monitor the behavior of the saving rate over time, which is likely to move towards either plus or minus infinity as $t$ increases.

(a) If the growth rate of savings moves towards plus (minus) infinity, then increase (reduce) $\rho$, calculate the new ratio of marginal utility to price in year 0, and repeat Step 3.

(b) If the growth rate of savings does not seem to converge, then the time frame is too short. Increase $T$ in Step 1 and repeat Steps 3 and 4.
(c) If the growth rate of savings seems to have either converged to a constant value or seems to be on a path towards convergence, increase $T$ in Step 1 and repeat Steps 3 and 4 to ensure that the apparent convergence is not spurious. The algorithm can easily be implemented by embedding the standard mechanism to solve a sequence of single-period models within a non-linear optimization routine that determines $\rho$, so that the change in the growth rate of savings in later years is below some predetermined level of tolerance.

Models with multiple agents can be solved as follows. If it is acceptable to assume a Social Welfare Function, then this function offers control over the aggregate saving rate by changing a single $\rho$ that applies to all agents (see, for example, Yang, 1999). The algorithm therefore does not need to be modified. In applications in which a Social Welfare Function is not meaningful (for example, in international trade models), it is necessary to assume that the $n$ different agents have distinct values of $\rho_1, \ldots, \rho_n$. The standard solution mechanism for a sequence of single-period models is then embedded within a multi-dimensional optimization routine (for example, Newton-Raphson type) that simultaneously determines the values of $\rho_1, \ldots, \rho_n$ that lead to the convergence of each agent’s savings growth rate.

In applications with many unknowns per single period, it may still be too time-consuming to solve the model repeatedly for enough years for the growth rate of savings to have converged to a constant. As an alternative to choosing a large value of $T$, one can follow the suggestion of Lau et al. (2002) to solve the model only until the growth rate of savings equals the growth rate of output, which may occur well before both have reached their steady state value.
The dynamic calibration algorithm is based on the fact that the observed level of consumption in year 0 permits the calculation of the marginal utility of consumption for different values of $\rho$. This strategy does not work for the counterfactual model where consumption in year 0 is determined as part of the dynamic optimization and $\rho$ is treated as a constant. It is therefore necessary to modify the algorithm slightly—instead of choosing the value of $\rho$, chose the ratio of marginal utility to price that leads to converge of the saving rate for a given $\rho$.

3.2 Algorithm to Determine the Solution of the Counterfactual Dynamic Model

**Step 1:** Choose the number of years for which to solve the model.

**Step 2:** Chose an initial estimate of the ratio of marginal utility of consumption to the consumption price index in year 0 (for example, the ratio used in the calibration stage).

**Step 3:** Use equation (4a) to determine consumption in year $t$, $t = 0, \ldots, T$. The only difference to Step 3 in the calibration algorithm is that it is now necessary to determine $C_0$ and $p_0$ as well.

**Step 4:** Monitor the behavior of the agent’s saving rate over time.

(a) If the growth rate of savings moves towards plus (minus) infinity, then increase (reduce) the estimate of the marginal utility to price and repeat Step 3.

(b) If the growth rate of savings does not seem to converge, then the time frame is too short. Increase $T$ in Step 1 and repeat Steps 3 and 4. It will also be necessary to analyze additional periods in the base case (which can be done
by solving the base case for additional periods; there is no need for new calibration) to be able to assess the changes in the years beyond the original value of $T$.

(c) If the growth rate of savings seems to have either converged to a constant value or seems to be on a path towards convergence, increase $T$ in Step 1 and repeat Steps 3 and 4 to ensure that the apparent convergence is not spurious.

3.3. Application

We illustrate our algorithm with the model described in Tideman et al. (2002), which was solved using the algorithm. The model uses the behavior of an infinitely-lived representative agent in a closed economy to analyze the effects of alternative tax policies in the United States. One of the tax policies examined is the substitution of a flat income tax for all federal, state, and local income taxes (see, for example, Hall and Rabushka, 1995). The model traces the effects of this change in taxes on wages, work effort, return to capital, saving, consumption, and welfare. The intertemporal elasticity of substitution is set equal to 0.375, and the base case is solved by calibrating the pure rate of time preference to be the rate that leads to a path of saving with a horizontal asymptote. The calibrated path of the social saving rate from 2000 to 2042 is shown as the solid line in Figure 1. The pure rate of time preference that produces a horizontal asymptote is 1.0511% per year. With a higher pure rate of time preference, the effort to maintain the ratio in 2000 of the marginal utility of consumption to the price of consumption produces a path of saving turns downward and heads toward $-\infty$. With a lower pure rate of time preference, the path of saving turns upward and heads for $\infty$. 
The counterfactual analysis assumes that the flat tax was introduced in 2002 and that this had not been anticipated. This changes behavior beginning in 2002. Where the ratio of the marginal utility of consumption to the price of consumption in the base case had been 7.2515, the counterfactual case requires a ratio of 7.2670 to achieve a horizontal asymptote for the path of saving. This is shown as the dashed line in Figure 1. The model suggests that the increase in saving that would result from the introduction of a flat tax would average 2.7% of NNP in the first 40 years.

4. Conclusion

Many dynamic CGE models describe the behavior of agents who live forever and have perfect foresight over an infinite number of periods. To be able to solve these
models, existing algorithms either assume that agents have perfect foresight over only one period (which does not correspond to rational expectations) or solve the model for a fixed number of periods and introduce a more or less arbitrary closing condition into the last period. We have described a simple and intuitive algorithm that permits the sequential solution of dynamic CGE models with agents who have perfect foresight over an infinite number of periods and which does not require any assumptions about behavior in a “final” period. The algorithm is easy to implement and eliminates the need to model non-rational expectations in cases when there is little empirical reason to deviate from rational expectations.
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