Abstract

We examine determinacy and expectational stability (learnability) of rational expectations equilibrium (REE) in sticky price “New Keynesian” (NK) models of the monetary transmission mechanism. We consider three different New Keynesian models: a labor-only model and two models that add capital – one where capital is allocated in an economy-wide rental market and another that supposes that the demand for capital is firm-specific. We find that Bullard and Mitra’s (2002, 2006) findings on determinacy and learnability of REE under various interest rate rules in the labor-only NK model do not always extend to models with capital. In particular, the Taylor principle, that the response of interest rates should be more than proportionate to changes in inflation, will not generally suffice to guarantee determinate and/or learnable equilibria in NK models with capital.

JEL Codes: D83, E43, E52.

Keywords: Learning, E-stability, monetary policy, New Keynesian model, capital, firm-specific capital.
1 Introduction

Taylor’s principle, that stabilizing rule-based monetary policy requires a more-than-proportional rise in the central bank’s target interest rate in response to higher inflation, has attracted considerable attention in the monetary policy literature. Taylor himself (1999) has shown how violations of this principle appear to explain historical episodes of high inflation and low capacity utilization. Woodford (2001, 2003a) has shown that Taylor’s principle ensures determinacy of rational expectations equilibria (REE) in New Keynesian models, i.e., dynamic, stochastic general equilibrium models with imperfect competition and Calvo-style staggered price setting. Bullard and Mitra (2002, 2006) show that Taylor’s principle further implies that the REE of New Keynesian models are learnable by agents who do not initially possess rational expectations. Our aim in this paper is to revisit the determinacy and learnability issue in New Keynesian models that, unlike the papers referenced above, also include investment decisions and endogenous capital accumulation.

A REE is determinate if the solution path is locally unique, thereby allowing application of standard comparative static exercises and preventing non-fundamental “sunspot” variables from playing any role. Such determinacy is desirable in that it enables policymakers to correctly anticipate the impact of policy changes (e.g. to the interest rate target) on inflation and output. A REE is learnable or “expectationally stable” in the sense of Evans and Honkapohja (2001) if agents who do not initially possess rational expectations are able to learn that REE using an adaptive real-time updating process such as recursive least squares. The conditions under which a REE is determinate will generally differ from the conditions under which that same REE is learnable. Satisfaction of both conditions is a desideratum of good monetary policy. In this paper we examine the conditions under which REE of the New Keynesian model with capital are both learnable and determinate under five different versions of Taylor’s rule that have appeared in the literature.

It is important to add capital to forward-looking, New Keynesian, sticky price models as such models are more general than the more widely studied, two equation “labor-only” version of that model. In that model, both inflation and output are non-predetermined “jump” variables (in the language of Blanchard and Kahn (1980)); in the absence of serially correlated shocks, the forward looking system would not display any dynamic persistence at all. The addition of capital—a pre-determined, non-jump variable—to the New Keynesian model thus provides for endogenous dynamics and greater persistence. It also allows for an analysis of investment decisions, an important and volatile component of aggregate demand. As Woodford (2003a, p. 352) notes, “one may doubt
the accuracy of the conclusions obtained [using the simple labor–only model], given the obvious
importance of variations in investment spending both in business fluctuations generally and in the
transmission mechanism for monetary policy in particular.” Indeed, in models with capital, central
banks’ influence on real interest rates may be greatly reduced as the arbitrage-induced equivalence
between the real interest rate and the marginal product of capital places an additional constraint
on interest rate movements that is not present in the labor-only model.

Two approaches have been taken to adding capital to New Keynesian models. The first, and
perhaps most straightforward approach, involves adding an economy-wide rental market for the
capital stock. The capital good is demanded by firms for use in combination with labor to produce
output and is free to flow to any firm in the economy in response to firms’ demands for the capital
good. A second approach, advocated by Woodford (2003a, Chp. 5; 2005) and developed further
by Sveen and Weinke (2005, 2006), imagines that capital is firm-specific; once the capital good has
been purchased for use by a specific firm, that capital cannot be reallocated for use by other firms
unlike in the economy-wide rental market model. Woodford’s purpose in proposing this firm-specific
model of capital was to make price adjustment by those firms who are free to adjust prices (under
the standard Calvo pricing assumption) less rapid – more sticky – by comparison with the rental
market model of capital. Indeed, an advantage of the firm-specific model of capital is that it does
not require an unrealistically high degree of price stickiness to match empirical facts, as is the case
in the rental market formulation. In our analysis we make use of both the rental and firm-specific
models of capital and we compare determinacy and learnability findings in these models with the
more widely studied labor-only model.

Our main contribution is to show that prior findings regarding both determinacy and learnability
of REE under a labor-only New Keynesian model of the economy may be altered by the addition
of investment decisions and endogenous capital formation. In particular we show that for the firm-
specific model of capital, the Taylor principle no longer suffices for determinacy and learnability of
REE under any of the policy rules we consider. In the rental market model of capital, the Taylor
principle suffices for determinacy and learnability of REE under just one of the five policy rules
we consider, though this finding appears specific to the calibration we choose. Thus in general,
the Taylor principle does not suffice to insure both determinacy and E-stability of REE in New
Keynesian models with capital in contrast to Bullard and Mitra’s (2002, 2006) findings for the labor-
only version of the New Keynesian model. At the same time, many of the policy implications that
emerge from our analysis of determinacy and learnability in models with capital reinforce some
of the more general conclusions of Bullard and Mitra (2002, 2006) regarding the type of policy rules that are most likely to insure determinacy and learnability of REE, the important role of policy-smoothing and the appropriate weighting of output. A second contribution of our approach is that we consider a wide variety of policy rules that have appeared in the literature together with three different New Keynesian models - the labor only model, the rental market for capital model and the firm-specific capital model, all under the same calibration of structural parameters, which serves to clearly highlight differences among these policy rules and models.

2 Related Literature

The determinacy and/or learnability of REE in New Keynesian models with capital has been explored in several recent papers. Dupor (2001) finds that the Taylor principle can induce an indeterminate REE in a continuous-time sticky price model with money and capital. Carlstrom and Fuerst (2005) show that Dupor’s finding is sensitive to the continuous time framework he uses; in a discrete-time variant of Dupor’s model, Carlstrom and Fuerst (2005) show that the Taylor principle can suffice to induce a determinate REE, in contrast to Dupor’s finding, provided that the interest rate rule depends on current inflation. On the other hand, if the interest rate rule depends on expected future inflation, Carlstrom and Fuerst show that, consistent with Dupor’s finding, the Taylor principle is not sufficient to implement a determinate REE and will almost always implement an indeterminate equilibrium. Carlstrom and Fuerst consider interest rate rules that respond only to inflation and not to output. Kurozumi and Van Zandweghe (2007) show that Carlstrom and Fuerst’s findings can be sensitive to the specification of the interest rate rule, in particular whether some weight is given to output. They show that, if the interest rate rule gives weight to expected future inflation as well as to current (and not expected future) output, then the Taylor principle will suffice to implement a determinate REE. Kurozumi and Van Zandeweghe (2007) go further and show that the conditions for determinacy of REE generally coincide with the conditions for E-stability or learnability of REE for the interest rate rules they consider. Xiao (2005) adds a small, empirically plausible amount of increasing returns to scale to a New Keynesian model with capital. With this addition, he shows that the Taylor principle no longer suffices to guarantee either determinacy or learnability of REE.

The discrete-time models of Carlstrom and Fuerst, Kurozumi and Van Zandeweghe and Xiao all suppose there is an economy-wide rental market for capital. Sveen and Weinke (2005) consider a discrete time version of the New Keynesian model without money but with firm-specific capital
and convex capital adjustment costs. They show that in this setting, the Taylor principle is not sufficient for determinacy of REE in the simplest case of an interest rate rule that responds to current inflation only. This finding stands in sharp contrast to Carlstrom and Fuerst’s findings for the New Keynesian model with a rental market for capital. Sveen and Weinke show that interest rate rules that respond to current inflation and output or which involve some policy smoothing are better able to induce determinate REE than is an interest rate rule that responds only to current inflation. They do not consider the E-stability of REE in the firm-specific capital case, and one aim of our paper is to explore this issue.

A general impression of this literature is that in New Keynesian models with capital, the Taylor principle need not suffice to insure determinacy of REE. However, much less is known about E-stability of these REE; with the exception of Xiao (2006) and Kurozumi and Van Zandeweghe (2005), no authors have explored the learnability of REE in New Keynesian models with capital, and the case of firm-specific capital has not been previously considered. The firm-specific case is typically modeled together with the assumption of convex capital adjustment costs which are not present in the rental-market case; we therefore add such adjustment costs to the rental market model to facilitate a comparison between that model and the firm-specific model of capital. More generally, comparisons of determinacy and learnability results between models with and without capital (labor-only) have not been made and there is not much consistency in the choice of interest rate rules and model calibrations used across the various studies. In this paper, we provide a thorough and consistent analysis of determinacy and learnability of REE in three versions of the New Keynesian model - the first generation “labor only” models and the second generation models with either a “rental market” for capital or firm specific capital. In addition, we consider the five main interest rate rules that have appeared in the literature: 1) a current data rule, 2) a lagged data rule, 3) a forward expectations rule, 4) a contemporaneous expectations rule and finally, 5) two versions of a policy smoothing rule. Some of our findings, e.g., E-stability of REE in New Keynesian Models with firm specific capital, are new while other findings are previously known, but in the latter case the value added of our paper lies in considering a consistent calibration and set of policy rules across all model specifications.¹ Our approach provides the reader with the clearest available picture to date of the conditions under which interest rate rules work to implement determinate and learnable REE in forward looking New Keynesian models of the monetary transmission mechanism.

¹Our analysis thus extends and encompasses that of Bullard and Mitra (2002, 2006).
3  A New Keynesian Model with Capital

3.1 The environment

We consider two different environments that differ in their treatment of capital. Our benchmark model is one involving an economy-wide rental market for capital. The alternative model has firm-specific capital. The labor-only model is shown to be a special case of both models with capital.

3.1.1 Rental market for capital

The economy is composed of a large number of infinitely-lived consumers. Each consumes a final consumption good \( C_t \), and supplies labor \( N_t \). Savings can be held in the form of real money balances \( \frac{M_t}{P_t} \), bonds \( \frac{B_t}{P_t} \), or capital \( K_t \). Consumers seek to maximize expected, discounted life-time utility:

\[
E_0 \sum_{t=0}^{\infty} \beta^t \left[ C_t^{1-\sigma} - \sigma N_t^{1+b} - v K_t^{1+\chi} \right],
\]

where \( \sigma, \gamma, b, v, \chi > 0 \) and \( 0 < \beta < 1 \). The budget constraint is given by

\[
C_t + \frac{M_t}{P_t} + \frac{B_t}{P_t} + I_t = \frac{W_t}{P_t} N_t + \frac{M_{t-1}}{P_t} + \frac{R_t}{P_t} K_t + (1 + i_{t-1}) \frac{B_{t-1}}{P_t} + D_t,
\]

and investment is defined as

\[
I_t = I \left( \frac{K_{t+1}}{K_t} \right) K_t.
\]

The consumer’s sources of income are its real labor income \( (W_t/P_t) N_t \), real money holdings \( M_{t-1}/P_t \) carried over from period \( t - 1 \), real capital rental income \( (R_t/P_t) K_t \), its real return on one-period bonds \( B_{t-1} \) purchased in period \( t - 1 \) and earning a gross nominal return of \( 1 + i_{t-1} \), and its dividends from ownership of firms, \( D_t \). The consumers allocate this income among consumption \( C_t \), new real money holdings \( M_t/P_t \), new bond purchases \( B_t/P_t \), and new investment \( I_t \). To allow comparisons between this environment and the one with firm specific capital (described below), we follow Woodford (2003) and suppose that each firm faces capital adjustment costs. Denote these costs by \( I(\frac{K_{t+1}}{K_t}) \), where the function \( I(\bullet) \) is assumed to satisfy the steady state conditions: \( I(1) = \delta, I'(1) = 1, \) and \( I''(1) = \varepsilon \psi \). Here, \( 0 < \delta < 1 \) denotes the depreciation rate and \( \varepsilon \psi > 0 \) characterizes the curvature of the adjustment cost function.\(^2\)

The first order conditions for the consumer’s problem can be written as:

\(^2\)The parameter \( \varepsilon \psi \) has been interpreted as the elasticity of the investment/capital ratio with respect to Tobin’s \( q \), in the steady state.
\[ vN_t^\chi = C_t^{-\sigma} \frac{W_t}{P_t}, \]  

(3)  

\[ C_t^{-\sigma} = \gamma \left( \frac{M_t}{P_t} \right)^{-b} + \beta E_t C_{t+1}^{-\sigma} \frac{P_t}{P_{t+1}}, \]  

(4)  

\[ 1 = \beta E_t \left( \frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}}(1 + i_t), \]  

(5)  

\[ \frac{dI_t}{dK_{t+1}} = E_t \left( \frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{R_{t+1}}{P_{t+1}} - \frac{dI_{t+1}}{dK_{t+1}}. \]  

(6)  

There exists a continuum of monopolistically competitive firms producing differentiated intermediate goods. The latter are used as inputs by perfectly competitive firms producing the single final good.

The final good is produced by a representative, perfectly competitive firm with a constant returns to scale technology

\[ Y_t = (\int_0^1 Y_{jt}^{\frac{\varepsilon-1}{\varepsilon}} dj)^{\frac{1}{1-\varepsilon}}, \]  

(7)  

where \( Y_{jt} \) is the quantity of intermediate good \( j \) used as an input, and \( \varepsilon > 1 \) governs the price elasticity of individual goods. Profit maximization yields the demand schedule

\[ Y_{jt} = (\frac{P_{jt}}{P_t})^{-\varepsilon} Y_t, \]  

(8)  

which, when substituted back into (7), yields

\[ P_t = \left( \int_0^1 P_{jt}^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}}. \]  

(9)  

The intermediate goods market features a large number of monopolistically competitive firms. The production function of a typical intermediate goods firm is:

\[ Y_{jt} = K_{jt}^{\alpha} N_{jt}^{1-\alpha}, \]  

(10)  

where \( K_{jt} \) and \( N_{jt} \) represent the capital and labor services hired by firm \( j \).

These firms’ real marginal cost \( \varphi_{jt} \) is derived by minimizing costs:

\[ \varphi_{jt} = \frac{1}{(1-\alpha)} \frac{W_t N_{jt}}{P_t Y_{jt}} = \frac{1}{\alpha} \frac{R_t K_{jt}}{P_t Y_{jt}}. \]  

(11)  

From this we can derive the expression

\[ \frac{K_{jt}}{N_{jt}} = \frac{\alpha}{1-\alpha} \frac{W_t}{R_t}, \]  

(12)
which implies that the capital-labor ratio is equalized across firms, as is marginal cost itself.

Intermediate firms set nominal prices in a staggered fashion, according to the stochastic time dependent rule proposed by Calvo (1983). Each firm resets its price with probability $1 - \omega$ each period, independent of the time elapsed since the last price adjustment and does not reset its price with probability $\omega$. A firm resetting its price in period $t$ seeks to maximize:

$$E_t \sum_{i=0}^{\infty} \omega^i \beta^t \left( \frac{C_{t+i}}{C_t} \right)^{-\sigma} \left( \frac{P_t^*}{P_{t+i}} Y_{jt+i} - \varphi_{jt+i} Y_{jt+i} \right),$$  

(13)

where $P_t^*$ represents the (common) optimal price chosen by all firms resetting their prices in period $t$. This maximization problem yields the first order condition

$$E_t \sum_{i=0}^{\infty} \omega^i \beta^t \left( \frac{C_{t+i}}{C_t} \right)^{-\sigma} Y_{jt+i} \left( \frac{P_t^*}{P_{t+i}} - \frac{\varepsilon}{\varepsilon - 1} \varphi_{t+i} \right) = 0.$$  

(14)

The equation describing the dynamics for the aggregate price level is

$$P_t = \left[ \omega P_t^{1-\varepsilon} + (1 - \omega) P_t^{*1-\varepsilon} \right]^{1/1-\varepsilon}.$$  

(15)

Finally, market clearing in the factor and goods markets implies that: $N_t = \int_0^1 N_{jt} dj$, $K_t = \int_0^1 K_{jt} dj$, $Y_t = \int_0^1 Y_{jt} dj$ and $C_t + I_t = Y_t$.

### 3.1.2 Firm-specific capital

Woodford (2003a, 2005) proposes a different version of the New Keynesian model in which an economy-wide rental market for capital does not exist. Instead, firms are assumed to accumulate capital for their own use only. This assumption implies that a firm’s price-setting decision is no longer separate from its capital accumulation decision, (as it is in the rental market case), and this change leads to important changes in the dynamics of the New Keynesian model with capital. The main advantage of the firm-specific approach to capital accumulation is that it does not require an unrealistically high degree of price stickiness to match empirical facts relative to the New Keynesian model with economy-wide rental markets that was examined in the previous section.

With firm-specific capital, the model needs to be modified as follows. First, the consumer’s budget constraint (1) is restated as:

$$C_t + \frac{M_t}{P_t} + \frac{B_t}{P_t} = \frac{W_t}{P_t} N_t + \frac{M_{t-1}}{P_t} + (1 + i_{t-1}) \frac{B_{t-1}}{P_t} + D_t.$$  

(16)

That is, consumers no longer make investment decisions given the absence of any economy-wide capital market.
Second, the firm’s problem is now defined as

\[
\max \sum_{i=0}^{\infty} \beta^i \left( \frac{C_{t+i}}{C_t} \right)^{-\sigma} \left( \frac{P_{jt+i}}{P_{t+i}} Y_{jt+i} - \frac{W_{t+i} N_{jt+i}}{P_{t+i}} - I_{jt+i} \right)
\]

subject to constraint (8), (10), where firm-specific investment is given by:

\[
I_{jt} = I \left( \frac{K_{jt+1}}{K_{jt}} \right) K_{jt},
\]

Notice that investment demand (18) is in the same form as (2), (i.e., it involves the same convex adjustment function \(I(\bullet)\)) but here it is firm-specific. Note also that \(P_{jt+i+1} = P_{jt+i}\) with probability \(\omega\).

Most first order conditions, such as (3), (4), and (5), continue to hold in the New Keynesian model with firm-specific capital. However, three differences between this setup and the rental-market setup will eventually lead to differences in the dynamics of the model.

First, the first order condition associated with capital is different in the firm-specific capital model than in the rental-market for capital model. Maximizing (17) with respect to capital yields:

\[
\frac{dI_{jt}}{dK_{jt+1}} = E_t \left( \frac{C_{t+1}}{C_t} \right)^{-\sigma} \left( \frac{MS_{jt+1}}{P_{t+1}} - \frac{dI_{t+1}}{dK_{t+1}} \right),
\]

where \(MS_{jt+1}\) denotes the nominal reduction in firm \(i\)'s labor cost associated with having an additional unit of capital in period \(t+1\), and is derived from the firm’s maximization problem as

\[
MS_{jt} = W_t \frac{MPK_{jt}}{MPL_{jt}},
\]

where MPK and MPL represent the marginal product of capital and of labor, respectively, of firm \(j\).

Second, marginal cost is now derived from the firm’s maximization problem as:

\[
\varphi_{jt} = \frac{W_t}{P_t MPL_{jt}}.
\]

The critical feature here is that marginal costs are no longer equalized across firms. They depend on each firm’s specific level of capital and labor.

Third, the first order condition associated with \(P_{jt+i}\) looks identical to (14), but after substituting in the expression for marginal cost, pricing decisions become a function of firm-specific capital. Since a firm’s marginal cost is affected by its current and future capital levels, its pricing decisions must also depend on its current and future capital levels. Future capital levels, on the other hand, depend in turn on today’s price and the future prices set by the firm. This complicated mechanism is absent in the rental-market case. Woodford (2005) shows that a linearized inflation equation can be computed by applying the method of undetermined coefficients.
3.1.3 Labor Only Model

For comparison purposes, we also study a version of the model in which labor is the only input in production. Setting \( I = K = 0 \) in our benchmark case will reduce the model to a generic, labor-only New Keynesian model. We assume production has constant returns to scale in labor:

\[
Y_{jt} = N_{jt}.
\]

The consumer’s budget constraint is the same as (16), and the economy wide resource constraint is simply \( Y_t = C_t \). The key first order conditions are (3), (4), (5) and (14).

3.2 Reduced linear systems

In the next three subsections we describe the system of linearized equations we use in our analysis of the determinacy and E-stability of REE in each of the three models that we consider. We use lower case letters to denote percentage deviations of a variable from its steady state value.

3.2.1 Benchmark Model: Rental market for capital

In the benchmark model with a rental market for capital, there are six non-dynamic equations and four dynamic equations. The first equation is the linearized version of the labor supply schedule (3):

\[
\chi n_t + \sigma c_t = w_t - p_t.
\]

The second and third equations are the linearized versions of (11). We are interested in the average level of marginal costs, which are given by

\[
\varphi_t = n_t + (w_t - p_t) - y_t,
\]

\[
= k_t + (r_t - p_t) - y_t.
\]

The fourth equation is the linearized production function

\[
y_t = \alpha k_t + (1 - \alpha)n_t.
\]

The first dynamic equation is New Keynesian Phillips curve, which is derived by solving the firm’s dynamic price-setting problem and combining it with (15). This equation is given by

\[
\pi_t = \beta E_t \pi_{t+1} + \kappa \varphi_t,
\]

where \( \kappa = \frac{(1-\omega)(1-\beta\omega)}{\omega} \).
The second dynamic equation is the linearized version of (6), which describes the evolution of capital:
\[
\Delta k_{t+1} = \beta E_t \Delta k_{t+2} + \frac{1}{\psi} \left\{ [1 - \beta (1 - \delta)] E_t (r_{t+1} - p_{t+1}) - (i_t - E_t \pi_{t+1}) \right\}.
\] (26)

The third dynamic equation is the Euler equation (5), which can be linearized as
\[
c_t = E_t c_{t+1} - \frac{1}{\sigma} (i_t - E_t \pi_{t+1}).
\] (27)

The last dynamic equation is the market clearing condition
\[
y_t = \frac{C}{Y} c_t + \frac{K}{Y} [k_{t+1} - (1 - \delta) k_t],
\] (28)
where \( C, I \) and \( Y \) represent steady state levels of consumption, investment and output.

Finally, we add the interest rate rule and use the non-dynamic equations to substitute out seven variables \( k^*_t = \Delta k_{t+1}, \ w_t - p_t, \ r_t - p_t, \ x_t, \ i_t, \ \varphi_t, \) and \( y_t \). The system becomes a four dimensional linear difference equation system consisting of \( s_t = (c_t, n_t, k_t, \pi_t)' \):
\[
E_t s_{t+1} = Js_t.
\] (29)

### 3.2.2 Firm-specific capital

With firm-specific capital, the inflation equation becomes
\[
\pi_t = \beta E_t \pi_{t+1} + \kappa^* \varphi_t,
\] (30)
which looks quite similar to (25), but the parameter \( \kappa^* \) is different from the parameter \( \kappa \) in (25).

Woodford (2005) develops an algorithm that utilizes the method of undetermined coefficients to compute \( \kappa^* \). Sveen and Weinke (2006) show that \( \kappa^* \) can be approximated by \( \frac{1 - \frac{\alpha}{1 - \alpha + \alpha \varepsilon}}{1 - \alpha + \alpha \varepsilon} \kappa \). In our analysis we make use of this approximation. Recall that \( 0 < \alpha < 1 \) is capital’s share of output and \( \varepsilon > 1 \) governs the price elasticity of individual goods. Thus using the approximation, we have that \( \kappa^* < \kappa \), so that inflation is less responsive to changes in marginal costs in the firm specific model of capital as compared with the rental market model of capital. That is, as Sveen and Weinke (2005) point out, for any value of the Calvo sticky price parameter \( \omega \), prices will be stickier in the firm-specific model of capital than they will be in the rental market for capital model.

The marginal return to capital can be derived from (20) as
\[
ms_t = w_t - p_t + n_t - k_t,
\]
and the aggregate capital accumulation equation is a linearized version of (19):

$$
\Delta k_{t+1} = \beta E_t \Delta k_{t+2} + \frac{1}{\delta^2} \{ [1 - \beta (1 - \delta)] E_t m_{st+1} - (i_t - E_t \pi_{t+1}) \}.
$$

As in the rental-market for capital case, the model with firm-specific capital can be reduced to a four-dimensional linear system of expectational difference equations with the same variables as in (29).

### 3.2.3 Labor-only model

The labor-only New Keynesian model can be reduced to the New Keynesian Phillips curve and the expectational IS curve:

\[
\begin{align*}
\pi_t &= \beta E_t \pi_{t+1} + (\sigma + \chi) \kappa y_t, \\
y_t &= E_t y_{t+1} - \frac{1}{\sigma} (i_t - E_t \pi_{t+1}),
\end{align*}
\]

plus the first order condition (4).

### 3.3 Monetary Authority

The central bank sets the nominal interest rate \( i_t \) every period according to a simple linear rule contingent on information about output and inflation. Following Bullard and Mitra (2002, 2006), we consider five variants of the interest rate rule. The first variant is the “contemporaneous data” rule:

\[
i_t = \tau_\pi \pi_t + \tau_y y_t,
\]

where \( \tau_\pi \geq 0 \) and \( \tau_y \geq 0 \), and \( i_t, \pi_t \) and \( y_t \) denote percentage deviations of the interest rate, the inflation rate, and output from their steady state values. This is a version of Taylor’s original (1993) rule that conditions the interest rate on current output and inflation.\(^3\) The “Taylor principle” is that \( \tau_\pi > 1 \), or that interest rate changes should be more than proportional to changes in inflation.

A second rule that is commonly considered (e.g., by Clarida et al. 1999), is the “forward expectations rule”:

\[
i_t = \tau_\pi E_t \pi_{t+1} + \tau_y E_t y_{t+1},
\]

\(^3\)Taylor–type interest rate rules typically condition on inflation and output gaps, i.e., deviations of inflation from a target level and of output from potential output, rather than on the levels of these variables by themselves. As the determinacy/learnability conditions of the systems we consider depend only on the magnitudes of the coefficients impacting on inflation and output levels, we choose to work with interest rate rules such as (31) (as well as the four other types of rules that follow) which condition on these levels only; of course, all of our findings will continue to apply to rules that condition on inflation and output gaps.
where policy makers use expectations of future inflation and output using information available at
time \( t \) to determine the current interest rate target.

Since current data for output and inflation may not be available at time \( t \), some have suggested
restricting attention to the use of time \( t - 1 \) data on output and inflation in the determination of
the interest rate target. This consideration gives rise to the next two rules we consider. The third
rule is the “lagged data” rule, which may be seen as an alternative to the current data rule (31), is:

\[
i_t = \tau \pi_{t-1} + \tau_y y_{t-1}.
\]

Similarly, the fourth rule we consider, the “contemporaneous expectations” rule may be seen as an
alternative to the forward expectations rule (32) and is given by:

\[
i_t = \tau \pi_{t-1} E_{t-1} \pi_t + \tau_y E_{t-1} y_t,
\]

where policy depends on forecasts of output and inflation that are formed using data available
through time \( t - 1 \).

In addition to the above four rules, we also consider an interest rate smoothing rule, where the
policy maker gives some weight \( \rho \) to past interest rates and remaining weight \( 1 - \rho \) to the predictions
of an interest rate rule such as rules 1-4 given above. Policy smoothing rules have been considered
by Bullard and Mitra (2006) for the labor-only model; results for the two models with capital have
not been previously examined.

4 Methodology and Calibration

4.1 General Methodology

We now turn to our analysis of the determinacy and E-stability of REE under the three models
and five different interest rate rules. When we study E-stability properties, we focus only on REE
that are determinate.\(^4\) We use the benchmark model to explain our general methodology and leave
the computational details for each different rule to the Appendix.

The determinacy of REE is assessed by computing the eigenvalues of the system (29). Since
there is only one predetermined variable \( k_t \) and the system is of dimension four, the REE will be
determinate in this case if the number of explosive roots is three and the number of stable roots
is one (see Blanchard and Kahn (1980)). If the number of stable roots exceeds one, we have an
indeterminate REE. If there is no stable root, the system is explosive.

\(^4\) For an analysis of the E-stability properties of indeterminate rational expectations equilibria, see, e.g., Honkapohja
To study adaptive learning, we re-write the system as

\begin{align}
  b_zz_t + b_kk_t &= d_kE_tk_{t+1} + d_zE_tz_{t+1}, \
  k_{t+1} &= e_zz_t + e_kk_t,
\end{align}

where the second equation is derived from the capital accumulation equation, which does not involve any expectations and so does not need to be learned. We assume that agents use the perceived law of motion (PLM)

\begin{align}
  z_t &= a_1 + \psi k_t, \
  k_t &= a_2 + mk_t,
\end{align}

which is in the same form as the minimal state variable (MSV) RE solution. By contrast with RE, learning agents do not initially know the parameter vectors \(a_1, a_2, m\) and \(\psi\) and must learn these over time. Given the PLM, we calculate the forward expectations as

\begin{align}
  E_tk_{t+1} &= a_2 + mk_t, \
  E_tz_{t+1} &= a_1 + \psi E_tk_{t+1} = a_1 + \psi a_2 + \psi mk_t.
\end{align}

Substituting these expressions into (35), we obtain a T-mapping from \((a_1, a_2, \psi, m)'\) to the actual law of motion of the model. Following Evans and Honkapohja (2001), we say the REE is E-stable (learnable by adaptive agents) if the differential equation, \(\frac{d}{dt}(a_1, a_2, \psi, m) = T(a_1, a_2, \psi, m) - (a_1, a_2, \psi, m)\), evaluated at the REE solution, is stable. This condition requires that all eigenvalues of \(D[T(a_1, a_2, \psi, m) - (a_1, a_2, \psi, m)]\) evaluated at the REE have real parts that are less than zero. Evans and Honkapohja provide conditions under which this differential equation approximates the limiting behavior of the recursive algorithms that characterize adaptive agent learning.

It is worth pointing out that assumptions about the agents’ information set can be crucial in assessing E-stability results. In the baseline case outlined above, we implicitly assume that both the private sector and the central bank can observe current values of the variable \(k_t\). They use this information to obtain forecasts \(E_tz_{t+1}\) and \(E_tk_{t+1}\), which in turn determine the current values of \(z_t\). This assumption applies in models using the current data rule or the forward expectation rule. However, this assumption is sometimes criticized as being unrealistic, since current data are usually not available to economic agents.\(^5\) An alternative assumption is to assume that the agents

\(^5\)The case with the current data rule is especially controversial. As pointed out by Bullard and Mitra (2002), it implies that the central bank has “superior information” in that it reacts to current values of \(y_t\) and \(\pi_t\) while the private sector does not possess such information.
can observe current exogenous variables but only lagged values of the endogenous and state variables at time $t$. We apply this assumption in models using the lagged data rule or the contemporaneous expectations rule. Both the central bank and the private sector are assumed to have symmetric knowledge of the lagged data. With these assumptions, we derive the specific E-stability conditions for each interest rate rule, and present them in the Appendix.

### 4.2 Calibration

Table 1 gives the calibration of model parameters we use in all of our analyses. This calibration is similar to that used by many other researchers in this literature—specifically, our calibration is the same one used by Sveen and Weinke (2005).

The reader may have noticed that we have excluded exogenous disturbance processes from all three versions of the New Keynesian model we have considered. This was in the interest of simplicity, as our determinacy and learnability findings do not depend in any way on the calibration of these shock processes. Similarly, as certain model parameters such as $b$, $\gamma$ and $v$ do not come into play in our analysis of determinacy and learnability, we do not provide calibrations of those parameters.

### 4.3 Determinacy and Learnability of REE under various interest rate rules

Ideally, we would like to provide analytic results concerning the determinacy and learnability of REE under various interest rate rules. Unfortunately, except in a few special cases, such as those studied by Bullard and Mitra (2002) and Carlstrom and Fuerst (2005), analytic results are not possible. The reason for this is simple: with the addition of capital, the dimension of the systems we are considering is either four or five and too complicated to reduce to a system that would allow for analytic findings. This situation necessitates that we adopt a numerical approach. Still, to the
extent possible, we will try to provide some intuition for our numerical findings.\textsuperscript{6}

Our approach is as follows. Holding the model calibration constant, we vary just two parameters, the weights $\tau_\pi$ and $\tau_y$, in the various interest rate rules. The ranges allowed for these weights cover all empirically relevant cases; in particular we search over a fine grid of values for $\tau_\pi$ between 0 and 5 and for $\tau_y$ between 0 and 4. We use an increment stepsize of .02. For each possible pair of weights $(\tau_\pi, \tau_y)$ in this grid, we check whether the eigenvalues satisfy the conditions for 1) determinacy and 2) E-stability. If both conditions are satisfied, we indicate this in the figures below using a blue color. If neither condition is satisfied, no color is used – the white regions in the figures below. We use a green color to mark weight pairs for which the REE is determinate but E-unstable. Finally, we use the color yellow to indicate weight pairs for which all roots are explosive (greater than one).

\subsection*{4.3.1 Current data rule}

Determinacy and E-stability findings using the current data rule (31) in the three models (labor only, rental market for capital, firm-specific capital) are shown in the three panels of Figure 1. Values of the monetary policy rule weight $\tau_\pi$ are indicated on the horizontal axis (labeled tp) and values of the monetary policy rule weight $\tau_y$ are indicated on the vertical axis (labeled ty) in these (and all subsequent) figures. Notice that under the current data rule, all three figures show only the color blue or no color (white). The reason for this finding is that in the case of the current data rule (31), if a REE is determinate, it is also E-stable in all three models. The coincidence of determinacy and learnability conditions in the labor-only and rental market for capital models under a current data rule is known from the work of Bullard and Mitra (2002) and Kurozumi and Van Zandweghe (2007), respectively.\textsuperscript{7} The difference in the regions of determinate REE in the firm-specific and rental market models of capital under a current data rule (as discussed in further detail below) has been pointed out by Sveen and Weinke (2005). However, the finding for the firm-specific model of capital, that determinate REE are also learnable, is a new finding of this paper.

In addition to the important observation that determinacy implies learnability in all three models with a current data policy rule (31), we further observe that the Taylor principle, $\tau_\pi > 1$,

\textsuperscript{6}The Matlab code we used in our numerical analysis is available on request.

\textsuperscript{7}Bullard and Mitra (2002) show that under the current data rule in the labor only model, REE will be both determinate and learnable provided that (in our notation) $\tau_\pi + \frac{1 - \delta}{\sigma + \chi} \tau_y > 1$, which they refer to as the “long-run” Taylor principle following Woodford (2003a, Chapter 4). Using our model calibration, it can be shown that this same inequality precisely characterizes the border between determinacy/E-stability and indeterminacy/E-instability in our Figure 1 for the labor-only model. Kurozumi and Van Zandweghe (2007) have similar analytic conditions for the rental-market model of capital, but their conditions are derived under an interest rate rule that gives weight to current output (or its components consumption, investment) and to future expected inflation, $E_t \pi_{t+1}$, – a rule that differs from the current data rule (31) that we consider here.
Figure 1: Determinacy and E-Stability Results Under the Current Data Rule.
suffices to insure both determinacy and learnability of REE in the labor-only and in the rental market for capital models, but does not suffice in the firm-specific model of capital. Notice that in the firm-specific capital model (bottom panel of Figure 1) there is a very small “sliver” where \( \tau_y \approx 0 \) and \( \tau_\pi > 1 \), where the REE is both indeterminate and unlearnable. Figure 2 provides a blown-up view of this region. The presence of this region of indeterminacy is consistent with Sveen and Weinke’s (2005) findings under a current data rule that gives zero weight to output. Indeed, they report that for various values of \( \omega \), measuring the stickiness of prices (which include our calibrated value \( \omega = .75 \)), the Taylor principle may not suffice to guarantee determinacy of the REE unless sufficient weight is given either to real activity or to lagged interest rates (as in a policy smoothing rule).

The intuition for the difference between the rental market and firm specific capital cases is, as explained by Sveen and Weinke (2005), that the rental market model of capital accumulation as opposed to the firm-specific model more readily masks a potential indeterminacy problem that is not possible in the labor-only model. The indeterminacy problem is that in models with capital, investment booms can become self-fulfilling events. Suppose an investment boom arises for some non-fundamental reason, e.g., due to the realization of a sunspot variable. The immediate effect of the increase in investment spending is to increase aggregate demand and each firm’s short-run marginal costs. However, the larger capital stock in place in subsequent periods implies an increase in productivity and lower marginal costs. A central bank adhering to the Taylor principle would respond to the increase in short-run inflation by raising interest rates but respond to the decrease in long-run inflation by lowering interest rates. Thus, depending on the horizon of the central bank, the net effect of an investment boom can be a decrease in real interest rates which would serve to rationalize the investment boom, making it a self-fulfilling event.

Notice that this self-fulfilling outcome should be more likely the more forward-looking is policy. But the likelihood of this indeterminacy outcome also depends on the degree to which prices are sticky. If prices are very sticky, which is captured by a high value for the parameter \( \omega \), the investment boom will lead to a much smaller rise in immediate short-run inflation making it more likely that long-term real interest rates will be unaffected or will even start to fall in support of the investment boom. The final piece of the puzzle is to note that for any given value of \( \omega \), the degree of price stickiness will be greater in the firm-specific model than in the economy-wide rental market for capital. Recall that the inflation equation in the benchmark rental market for capital model (25) differed from that of the firm specific model (30) only in the coefficient attached
to marginal costs, i.e., $\kappa > \kappa^*$ implying less price stickiness in the rental market relative to the firm specific model of capital for the same value of $\omega$. In the model with an economy-wide rental market for capital, indeterminacy only occurs for implausibly high values of $\omega$, but in the firm-specific model of capital, indeterminacy can be generated for empirically plausible values of $\omega$. For instance, Sveen and Weinke (2005) show that for the rental market model to generate the same equilibrium dynamics as the firm specific capital model with an empirically plausible parameterization of $\omega = .75$, (i.e., price adjustment on average, every four quarters) would require setting $\omega = .9$ in the rental market model—an empirically implausible value (price adjustment on average, every 10 quarters). Alternatively put, for the baseline calibration we use, $\omega = .75$, prices will be sufficiently flexible in the rental market model of capital to avoid the indeterminacy outcome when the Taylor principle holds, but the same will not be true in the firm-specific model of capital.\(^8\) As we shall see, this same “sliver of a region” of indeterminacy/E-instability in the firm-specific capital model arises under all four of the interest rate rules we consider that don’t involve policy smoothing.

Adding some policy inertia may work to eliminate this region of indeterminacy as will be shown later in the paper.\(^9\) Of course, a judicious (and empirically plausible) choice of policy rule weights will also ensure that the REE is determinate and learnable in all three models under a current data rule. For instance, Taylor’s (1993) original parameterization of the current data rule, $\tau_\pi = 1.5$ and $\tau_y = 0.5$ succeeds in implementing a determinate and learnable REE in all three models. The clear recommendation that follows from our findings using the current data rule is that the Taylor principle, in tandem with some positive weight being given to real activity (or possibly to lagged interest rates) will more reliably implement both a determinate and learnable REE in models with capital.

### 4.3.2 The forward expectations rule

Determinacy and E-stability results for the forward expectations rule (32) in the three models are shown in Figure 3. Under this rule, the Taylor principle does not suffice to insure determinacy and learnability of REE in any of the three models and there are large differences in the regions

---

\(^8\)Of course, for calibrations other than the one we consider, e.g. higher values for $\omega$, the small sliver of indeterminacy we observe for the firm-specific model of capital under the current data rule when $\tau_y \approx 0$ will also appear in the rental market model of capital under the current data rule so that the Taylor principle will not suffice to insure determinacy and learnability of REE for such calibrations.

\(^9\)Another way to get rid of this sliver of indeterminacy in the firm-specific capital model is to assume more flexible prices, e.g. values of $\omega$ that are closer to zero. However, Sveen and Weinke (2006) show that if wages are also modelled as being sticky, much lower values for $\omega$ (greater price flexibility) does not eliminate the indeterminacy problem.
giving rise to determinate and learnable REE across the three models. Specifically, the addition of capital either via an economy-wide rental market or via firm-specific demand leads to a big reduction in the region for which the REE is determinate and learnable relative to the labor-only case. An important observation from Figure 3 is that in models with capital, the weight assigned to output under a forward expectations rule that obeys the Taylor principle should neither be too aggressive nor too modest. Notice further that the weight regions giving rise to determinate, E-stable REE appear to be empirically plausible ones; Taylor’s (1993) calibration will again work to insure determinacy/learnability of the REE in all three models.

The fact that the forward expectations rule (32) makes it more likely (relative to the labor-only case) that REE is indeterminate and unstable under adaptive learning in the rental market for capital model is essentially known from the work of Carlstrom and Fuerst (2005) and Kurozumi and Van Zandweghe (2007). Our findings for the firm-specific model of capital are new, and our findings for the rental market model of capital differ somewhat from those of Carlstrom and Fuerst and Kurozumi and Van Zandweghe. Those authors show that the use of interest rate rules that

10 The finding that the Taylor principle does not suffice for both determinacy and learnability of REE in the labor-only model under the forward expectations rule was previously shown by Bullard and Mitra (2002). They also report that under the forward expectations rule, REE will be determinate and learnable only if the weight assigned to output \( \tau_y \) is not too great. The lowering of this upper bound for \( \tau_y \) in models with capital and the addition of a new lower bound for \( \tau_y \) are new findings of this paper.
condition on expectations of future inflation alone or on future output as well always result in an indeterminate and unlearnable REE in the rental market for capital model, whereas we find that there is a plausible determinate/learnable region in the rental market case (as well as in the firm-specific capital market case).

What accounts for our different finding? We begin by noting that the addition of investment to the New Keynesian model imposes an arbitrage relationship between the return on bonds and on physical capital. As Carlstrom and Fuerst (2005) show, if the policy rule is purely forward-looking, this arbitrage relationship will hinge entirely on future expected variables, which produces a zero eigenvalue in the system. This forces the only state variable, the capital stock, to become a jump variable, and this insures that the equilibrium will be indeterminate. A critical feature in the model we consider is the inclusion of capital adjustment costs. Capital adjustment costs make capital accumulation dependent on current and not just future capital; this makes the arbitrage relationship not entirely forward-looking and eliminates the zero eigenvalue, which, in combination with a purely forward looking interest rate rule will implement a determinate REE in certain cases, i.e., with sufficiently high costs of adjustment.

To establish that capital adjustment costs are responsible for our different determinacy/E-stability findings under the forward expectations rule, Figure 4 shows the consequences of varying these adjustment costs in the rental market model of capital (similar results obtain if we vary adjustment costs in the firm-specific model of capital). More precisely we vary the parameter governing the curvature of the capital adjustment cost function, $\psi$, in the three panels of Figure 4 from a value very close to 0–0.1 (i.e. no adjustment costs), to 1 and finally to 15 (note the change in the scale of the graph in the last case). Recall that our baseline calibration had $\psi = 3$ (c.f. Figure 4 with Figure 3, rental market case).

Figure 4 shows clearly that when $\psi$ is (essentially) zero, there are essentially no weight pairs for which the REE is both determinate and E-stable, consistent with the findings of Carlstrom and Fuerst (2005) and Kurozumi and Van Zandweghe (2007). However as $\psi$ is increased above zero,  

---

11 We included capital adjustment costs in the rental market as they are included as part of Woodford’s and Sveen and Weinke’s model of firm-specific capital, and we wanted to make the comparison between the two models as close as possible. Of course, the inclusion of capital adjustment costs is a standard practice in neoclassical investment theory. Carlstrom and Fuerst (2005) briefly discuss the addition of capital adjustment costs to the rental market for capital model they examine, and note that such adjustment costs may overturn their conclusions for forward-looking policy rules.

12 An alternative mechanism for achieving the same end, as pursued by Kurozumi and Van Zandweghe (2007), is to have a hybrid policy rule that conditions on future expected inflation but on current output or its components (consumption, investment).
Figure 3: Determinacy and E-Stability Results Under the Forward Expectations Rule.
Figure 4: Determinacy and E-Stability Results Under the Forward Expectations Rule In the Rental Market for Capital for Various Values of the Parameter $\varepsilon_\psi$ (0.1, 1, and 15) Characterizing the Degree of Adjustment Costs.
the determinacy/E-stability region increases as well which is consistent with the intuition we have provided: the increasing convexity of adjustment costs means investment becomes both more costly and more tied to the current level of the capital stock; as the latter variable is predetermined, it makes the indeterminacy (and E-instability) outcome less likely.

4.3.3 The lagged data rule

Results for the lagged data policy rule (33) in the three models are shown in Figure 5. In these figures, several different colors are now visible. As a reminder to the reader, the blue colored regions indicate weight pairs \((\tau_\pi, \tau_y)\) for which the REE is both determinate and learnable. The green colored region is determinate but not learnable, the yellow colored region is where all roots are explosive and no color (white) represents regions where REE is indeterminate.

In this case we observe that in all three models, the simple Taylor principle does not suffice to insure both a determinate and learnable REE. This finding, for the labor-only model only, was earlier reported by Bullard and Mitra (2002). The novel finding we report here is for the two
models with capital. The determinate and learnable parameter regions in the models with capital are considerably smaller relative to the labor-only model, suggesting that a much more modest response to output is needed for determinacy and learnability of the REE. In the firm-specific model of capital there is again a small sliver of indeterminate REE for values of $\tau_y$ that are close to 0 and values for $\tau_\pi$ between 1 and 3.

However, there is a sense in which the findings for the models with capital follow somewhat continuously from the labor-only model; if we increased the range of values of $\tau_y$ we considered above 4 in Figure (5) in the labor-only model, we would also have a green-colored region for the labor-only model where the Taylor principle was not satisfied, i.e. $\tau_\pi < 1$ and $\tau_y > 4$ and where the REE was determinate but E-unstable. That is, qualitatively, the pictures for all 3 models are quite similar.

One explanation for smaller region of determinacy and learnability in and the models with capital is the greater dimension of the system when there is capital and a lagged data policy rule – 5 equations – as compared with the labor-only model which has a dimension of just 3 equations under the lagged data policy rule (two lagged values of capital versus one). Thus the models with capital require that 2 out of 5 of the eigenvalues (equal to the number of predetermined variables) should be stable as opposed to just 1 out of 3 in the labor only model and 1 out of 4 in models with capital using current data, both of which are less restrictive requirements than in the models with capital and a lagged data rule.

Comparing the two different approaches to modeling capital, the firm-specific case leads to a slightly larger region of determinate and learnable REE, though the firm-specific case continues to have a sliver of a region where equilibrium is both indeterminate and E-unstable. Nevertheless, for reasonable parameterizations of the Taylor rule, for instance using Taylor’s original (1993) calibration of $\tau_\pi = 1.5$ and $\tau_y = .5$, determinacy and learnability of the REE are assured in all three models.

4.3.4 The contemporaneous expectations rule

Results for the contemporaneous expectations rule (34) in the three models are shown in Figure 6. This case yields results that at first glance appear to be quite similar to the current data rule (compare Figure 6 with Figure 1). Indeed, under the contemporaneous expectations rule the Taylor principle again suffices to implement a determinate and learnable REE in the labor-only model. However, by contrast with the current data rule, under the contemporaneous expectations rule, the
Taylor principle no longer suffices to insure both determinacy and learnability of REE in either model of capital. In both the rental and firm-specific models of capital, the determinacy conditions under the contemporaneous expectations rule are exactly the same as under the current data rule. The Taylor principle suffices to insure determinacy of REE in the rental market case, but in the firm-specific case, there is the same small sliver of a region where \(\tau_y \approx 0\) and \(\tau_\pi\) is between 1 and 3 for which the REE is indeterminate. However, under the contemporaneous expectations rule there is a further difference relative to the current data rule: in both the rental and firm-specific models of capital there is now a small sliver of a region (colored green) where \(\tau_y\) is close to 0 and \(\tau_\pi\) is between 1 and 1.5 (rental market) or between 1 and 3 (firm-specific) for which the REE is determinate but is not E-stable. Thus in the rental market model under contemporaneous expectations, the Taylor principle may suffice for determinacy of REE but it no longer suffices for E-stability of REE. In the firm-specific model under the contemporaneous expectations rule, the region of determinate-but-E-unstable REE is a very small green sliver (that is admittedly difficult to see) but which lies along the border between the indeterminate (white) and determinate (blue) regions in Figure (6). As Figure 6 shows, these regions of indeterminacy or E-instability can be easily avoided by setting \(\tau_y\) sufficiently high. Indeed we observe that there is again a very wide range of plausible calibrations, (e.g. Taylor’s (1993) choices \(\tau_\pi = 1.5\) and \(\tau_y = 0.5\)) that result in determinate and learnable REE in all three models under the contemporaneous expectations rule.

4.3.5 Interest rate smoothing rules

We next consider two variants of interest rate smoothing rules studied in the labor-only model by Bullard and Mitra (2006). The first version is a lagged data policy smoothing rule:

\[
i_t = \rho i_{t-1} + (1 - \rho) [\tau_\pi \pi_{t-1} + \tau_y y_{t-1}],
\]

(37)

where \(\rho \in (0, 1)\) is the weight given to the past interest rate target. The RHS term in square brackets is just the lagged data rule considered earlier.

The second variant is a forward expectations policy smoothing rule:

\[
i_t = \rho i_{t-1} + (1 - \rho) [\tau_\pi E_t \pi_{t+1} + \tau_y E_t y_{t+1}],
\]

(38)

where again the RHS term in square brackets is just the forward expectations rule considered earlier.

We focus on these two versions of policy smoothing rules as the lagged data and forward expectations rules without policy inertia were previously found to be the most troublesome in
Figure 6: Determinacy and E-Stability Results under the Contemporaneous Expectations Rule.
terms of implementing determinate and learnable REE in New Keynesian models with capital. We
know from the findings of Bullard and Mitra (2006) for the labor-only model that the addition
of policy inertia (interest rate smoothing) can work to enlarge the region of policy weights for
which a policy rule yields determinate and learnable REE; indeed, Bullard and Mitra (2006) show
in the labor-only model that for sufficiently large policy inertia, the Taylor principle suffices for
determinacy of REE.

Here we follow the convention in much of the literature on monetary policy rules (e.g. Rudebusch
(2002)) and imagine that the weight assigned to the lagged interest rate, \(i_{t-1}\), and to the prescription
of the policy rule [in square brackets] add up to unity; in this case the interest rate rule without
smoothing can be regarded as the special limiting case where \(\rho \to 0\). Woodford (2003b) has shown
how such a “partial adjustment” model of monetary policy inertia may result from optimizing
behavior on the part of the central bank.\(^{13}\) We add the choice of \(\rho = .5\) to our baseline calibration
(Table 1) for both policy smoothing rules, but we later explore the impact of changes in this
persistence parameter.

Determinacy and learnability results for the three models under the lagged data rule with policy
smoothing (37) and our baseline calibration are shown in Figure 7. We see that in this case, the
Taylor principle suffices to guarantee both determinacy and learnability of REE in the labor-only
model but not in the two models that include capital. Comparing Figure 7 with Figure 5 which
showed results for the lagged data rule without inertia \((\rho = 0)\), we observe that the addition of
policy inertia (specifically, \(\rho = 0.5\)) greatly enlarges the range of policy weights for which REE are
determinate and E-stable in both models with capital. Thus policy inertia, like a positive weight
attached to output, helps policymakers avoid indeterminacy and E-instability. As in the case of the
other rules, one can find a large range of empirically plausible values for the policy weights \((\tau_\pi, \tau_y)\),
for which the REE is both determinate and learnable.

Determinacy and learnability results for the three models under the forward expectations policy
smoothing rule (38) are shown in Figure 8. In this case, the Taylor principle does not suffice for
determinacy and learnability of REE in any of the three models, as was also true under the forward

\(^{13}\) Some authors e.g., Rotemberg and Woodford (1998), Giannoni and Woodford (2003) have derived optimal policy
rules where the coefficient on the lagged interest rate is greater than 1. However, such a super-inertial policy rule
appears to be at odds with estimated interest rate rules. For instance, using U.S. data, Amato and Labauch (1999)
estimate the current data rule (31) with the addition of a lagged interest rate (dependent) variable and report that
the unrestricted coefficient estimate on the lagged interest rate is always less than one. While we think it would be
of interest to consider super-inertial interest rate rules, a virtue of the partial adjustment model we examine is that
it requires just one additional parameter, \(\rho\), making it easier to see whether our findings without inertia generalize
to the addition of some inertia.
Figure 7: Determinacy and E-Stability Results Under the Lagged Data Policy Smoothing Rule.
expectations rule (32) without policy smoothing ($\rho = 0$). Comparing Figure 8 with Figure 3 we again observe that the addition of policy inertia ($\rho = 0.5$) greatly enlarges the range of policy weights for which REE are determinate and E-stable in the two models with capital. However it remains the case that in both models with capital, determinacy and E-stability of REE requires both the Taylor principle, together with a $\tau_y$ that is neither neither too small nor too large. Notice also that under rule (38) in the labor-only model there now appears to be a large (green) region involving reasonable policy weights (e.g., those suggested by Taylor (1993)) for which the Taylor principle is satisfied and the REE is determinate but unlearnable (E-unstable). This quite different (and surprising) finding for the labor-only model under rule (38) underscores the need to consider New Keynesian models with both labor and capital.

We next explore the sensitivity of our findings using policy smoothing rules to changes in the persistence parameter $\rho$. We focus on 1) the rental market case and 2) the lagged data policy smoothing rule (37), though similar results obtain for the firm-specific capital case and under the
Figure 9: Sensitivity Analysis for the Rental Market Model Under the Lagged Data Policy Smoothing Rule Showing how the Determinate and E-stable Region Varies with Changes in the Value of $\rho$

forward expectations policy smoothing rule (available on request). Figure 9 below shows that in the case of a rental market for capital, the region of weight pairs for which REE is both determinate and E-stable increases as $\rho$ increases. Specifically, the lines in Figure 9 show how the boundaries of the determinate and E-stable region increase with increases in $\rho$. For instance, the determinate and E-stable polygon for the baseline $\rho = 0.5$ is the same in Figure 9 as the blue determinate and E-stable polygon in Figure 7. As $\rho$ is lowered to 0.2, the upper bound to this determinate and E-stable region falls relative to the baseline case and as $\rho$ is raised to 0.75, the upper bound to the determinate and E-stable region rises relative to the baseline case as Figure 9 illustrates.

As similar finding obtains for the rental market if we use the forward expectations policy smoothing rule or if we use either policy smoothing rule in the firm-specific model of capital. The main finding from this analysis is that increasing persistence in policy (the value of $\rho$) in models with capital leads to an expansion in the region where equilibrium is both determinate and learnable.
5 Conclusions

We have studied determinacy and learnability of REE in 3 different New Keynesian models. The first model with labor only is a standard, benchmark model. The other two models add productive capital: one via an economy-wide rental market for capital and one via firm-specific demand for capital. The addition of capital to the New Keynesian model allows for the study of investment decisions and may serve to temper the efficacy of central bank policies, as movements in real interest rates are now affected by capital market activity.

Determinacy and learnability are two highly desirable properties for REE and it should be the aim of central banks to adopt interest rate policies that implement equilibria possessing both of these properties. While Bullard and Mitra (2002, 2006) found that the Taylor principle nearly always suffices for both determinacy and learnability of REE in the labor only model, the addition of capital to the New Keynesian model requires some further qualifications to this conclusion. In particular, we find that in the model with a rental market for capital, the Taylor principle continues to suffice to insure both determinacy and learnability of REE if the interest rate rule responds to current data on inflation and output. However the Taylor principle need not suffice for both determinacy and learnability if the interest rate rule responds to future or contemporaneous expectations of inflation and output or to lagged values of these variables or if the central bank uses a policy smoothing rule. Perhaps our most important finding is that in the model with firm-specific capital the Taylor principle never suffices to insure both determinacy and learnability of REE for the calibration we consider. There is always at least some small region of policy weights that satisfy the Taylor principle but for which the REE is neither determinate nor learnable. Our findings suggest that this region can be avoided by giving sufficient weight to output or by adopting a sufficiently strong policy smoothing stance or both.\footnote{Alternatively, it may be avoided if prices are sufficiently (but perhaps implausibly) flexible in the firm-specific model.}

Under the forward-looking policy rule (32) our results underscore the important role played by capital adjustment costs. As we have shown in either model of capital, in the absence of capital adjustment costs the forward-looking policy rule (32) results in REE that are always indeterminate and E-unstable. Introducing capital adjustment costs ties investment decisions to the current level of the current capital stock thus making determinacy and E-stability of the REE a possibility.

A difficulty with our analysis is that it is entirely numerical; analytic results for four or five dimensional systems are difficult to obtain, and the lack of analytic results inhibits our understanding
of the causal mechanisms. On the other hand, as we have noted, our numerical approach confirms many existing analytical findings (especially for the simpler, labor-only model) thus providing us with a high degree of confidence in these numerical results; at the same time, our numerical approach has enabled us to provide several new findings, especially in the case of firm-specific capital. Our adoption of a single model calibration facilitates comparisons across the three models and provides the clearest picture yet of the conditions under which various interest rate rules will yield determinate and learnable REE.

While the Taylor principle may not suffice to guarantee determinacy and learnability in all models considered, we can still reach several practical conclusions that should be of interest to central bankers. First, while the specific findings for the labor-only model do not generalize to models with capital and investment decisions, there appears to be considerable continuity in several of the broader policy recommendations. Specifically, two of the rules we consider, the current data rule and the contemporaneous expectations rule, fare the best in all three models in terms of admitting the largest possible regions of determinate and learnable REE. Second, the results for the firm-specific and rental models of capital suggest that there is high value to interest rates that obey both the Taylor principle and give some weight to output and/or to policy smoothing so as to avoid indeterminacy and instability under learning. Under a forward-looking policy rule, the response to output should not be too modest nor too aggressive, and under a policy-smoothing rule, the weight attached to past interest rates should not be too small. Finally, we note that some \((\tau_\pi, \tau_y)\) pairs succeed in implementing determinate and learnable REE in all models and for nearly all interest rate rules that we have considered. In particular, the parameterization proposed by Taylor (1993) – \(\tau_\pi = 1.5\) and \(\tau_y = 0.5\) – belongs to that class. Perhaps the empirical success of Taylor’s (1993) rule rests as much with the parameter values he chose as with the principle that bears his name.

Appendix

In this section we derive conditions for E-stability for all variants of the Taylor rule used in the text.

5.1 Current Data Rule and Forward Expectations Rule

The linearized system can be written in matrix form as
\[ b_z z_t + b_k k_t = d_k E_t k_{t+1} + d_z z_{t+1}, \]  
\[ k_{t+1} = e_z z_t + e_k k_t, \]  
\[ z_t = (c_t, n_t, \pi_t)' \]. The three equations involving agents' expectations are summarized by (39), while (40) is the capital accumulation equation that does not involve any expectations.

The perceived law of motion (PLM) is

\[ z_t = a_1 + \psi k_t, \]  
\[ k_t = a_2 + m k_{t-1}, \]  
which corresponds to the MSV solution of the model.

Given the PLM, the expectations can be derived as

\[ E_t k_{t+1} = a_2 + m k_t, \]
\[ E_t z_{t+1} = a_2 + \psi E_t k_{t+1} = a_1 + \psi a_2 + \psi m k_t. \]

Substituting these into (1) and (2), one gets

\[ k_{t+1} = e_z a_1 + (e_z \psi + e_k) k_t, \]
\[ z_t = b_z^{-1}(d_k a_2 + d_z a_1 + d_z \psi a_2) + b_z^{-1}(d_k m + d_z \psi m - b_k) k_t. \]

Therefore the T-mappings are

\[ T(a_1) = b_z^{-1}(d_k a_2 + d_z a_1 + d_z \psi a_2), \]
\[ T(a_2) = e_z a_1, \]
\[ T(m) = e_z \psi + e_k, \]
\[ T(\psi) = b_z^{-1}(d_k m + d_z \psi m - b_k). \]

In principle, these T-mappings can be solved to obtain the REE equilibrium. However, solving the T-mappings directly may sometimes yield multiple solutions, and we need to find among these solutions the unique one that is consistent with equilibrium determinacy. To avoid this complication, we solve the REE equilibrium by applying the Blanchard and Kahn (1980) algorithm, which always yields a unique REE.
The T-mappings for $\psi$ and $m$ form an independent system, and we start by computing the derivatives

\[
\begin{align*}
dT_m(m, \psi) &= 0, \\
&T_m(m, \psi) = e_z, \\
&T_m(\psi, m) = b_z^{-1}(d_k + d_\psi), \\
&T_\psi(\psi, m) = b_z^{-1}d_z m,
\end{align*}
\]

For E-stability, we require that the matrix

\[
\begin{pmatrix}
0 & e_z \\
b_z^{-1}(d_k + d_\psi) & b_z^{-1}d_z m
\end{pmatrix}
\]

has eigenvalues less than 1. Since $a_1 = (0, 0, 0)'$ and $a_2 = 0$, the corresponding derivatives are easy to compute:

\[
\begin{align*}
&T_{a_1}(a_1) = b_z^{-1}d_z \\
&T_{a_2}(a_1) = b_z^{-1}(d_k + d_\psi) \\
&T_{a_1}(a_2) = e_z \\
&T_{a_2}(a_2) = 0
\end{align*}
\]

For E-stability, we require the matrix

\[
\begin{pmatrix}
b_z^{-1}d_z & b_z^{-1}(d_k + d_\psi) \\
e_z & 0
\end{pmatrix}
\]

to have eigenvalues less than 1.

### 5.2 Lagged Data Rule

With this rule, the underlined assumption is that agents can only observe lagged variables (otherwise they could have used current data in the rule). As a result, the MSV solution must have a different form. There are now 4 state variables instead of one: $c_{t-1}, n_{t-1}, \pi_{t-1}$ and $k_{t-1}$.

To obtain the MSV solution, we first substitute

\[
i_t = \tau_\pi \pi_{t-1} + \tau_y y_{t-1}
\]

into (39), then we rewrite (40) as

\[
k_t = e_z z_{t-1} + e_k k_{t-1}.
\]

34
Combining (39) and (40), the whole system can then be rewritten as

\[ X_t = F E_t X_{t+1} + L X_{t-1}, \quad (43) \]

where \( X_t = (z_t, k_t)' \).

The PLM is

\[ X_t = a + \gamma X_{t-1}. \]

Substituting this into the expectations, we get

\[ E_t X_{t+1} = a + \gamma a + \gamma^2 X_{t-1}. \]

Substituting this expectation into (43), we have

\[ X_t = F(a + \gamma a) + (F \gamma^2 + L) X_{t-1}. \]

The T-mappings are

\[ T(a) = F(a + \gamma a), \]
\[ T(\gamma) = F\gamma^2 + L. \]

We need to evaluate the eigenvalues of

\[ DT(a) = F(I + \gamma), \]
\[ DT(\gamma) = \gamma' \otimes F + I \otimes F\gamma \]

to determine if the REE is E-stable.

5.3 \textbf{Contemporaneous expectations}

With this rule, the informational assumption is that agents can only observe lagged variables, and the expectations are made at time \( t-1 \) rather than time \( t \). As a result, the MSV solution must have a different form. To obtain it, we first substitute

\[ i_t = \tau_x E_{t-1} z_t + \tau_y E_{t-1} y_t \]

into the system, and re-write the resulting system as

\[ g_k E_{t-1} k_t + g_z E_{t-1} z_t + b_z z_t + b_k k_t = d_k E_t k_{t+1} + d_z E_t z_{t+1}, \quad (44) \]
\[ k_{t+1} = e_k k_t + e_z z_t. \quad (45) \]
The PLM is
\[ z_t = a + \gamma k_t, \]
\[ k_t = b + mk_{t-1}. \]

Agents’ forecasts become
\[ E_{t-1}z_t = a + \gamma k_t, \]
\[ E_{t-1}z_{t+1} = a + \gamma (b + mk_t), \]
\[ E_{t-1}k_t = b + mk_{t-1}, \]
\[ E_{t-1}k_{t+1} = b + mk_t. \]

Substituting this into (44) and (45), we get the T-mappings
\[ T(a) = b_z^{-1}(d_k b + d_z a + d_z \gamma b - g_z a), \]
\[ T(b) = e_z a, \]
\[ T(m) = e_z \gamma + e_k, \]
\[ T(\gamma) = b_z^{-1}(d_k m + d_z \gamma m - g_k - b_k - g_z \gamma). \]

For E-stability, we require the following two Jacobian matrices to have eigenvalues less than 1:
\[
\begin{pmatrix}
0 & e_z \\
b_z^{-1}(d_k + d_z) & b_z^{-1}(d_z m - g_z)
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
b_z^{-1}(d_z - g_z) & b_z^{-1}(d_k + d_z \gamma) \\
e_z & 0
\end{pmatrix}.
\]

### 5.4 Interest rate smoothing

The lagged data rule with policy inertia, is given by:
\[ i_t = \rho i_{t-1} + (1 - \rho)(\tau_{\pi t-1} + \tau_{y t-1}). \]

Substituting in this rule, the system can be written as:
\[ X_t = F E_t X_{t+1} + L X_{t-1}, \tag{46} \]
where \( X_t = (z_t, k_t, i_t)' \).
The PLM and the critical matrices are the same as in the lagged data case, except that there is an extra state variable \( i_{t-1} \) in the solution.

The forward-looking rule with policy inertia is given by:

\[
i_t = \rho i_{t-1} + (1 - \rho)(\tau_\pi E_t \pi_{t+1} + \tau_y E_t y_{t+1}),
\]

Substituting in this rule, the system becomes:

\[
\begin{align*}
    b_z z_t + b_s s_t &= d_s E_t s_{t+1} + d_z E_t z_{t+1}, \\
    s_{t+1} &= e_s s_t + e_z z_t + f_z E_t z_{t+1} + f_s E_t s_{t+1},
\end{align*}
\]

where \( z_t \) is defined as before and \( s_t = (k_t, i_{t-1})' \). The PLM is

\[
\begin{align*}
    z_t &= a_1 + \psi s_t, \\
    s_{t+1} &= a_2 + ms_t.
\end{align*}
\]

Following the same procedures as explained above, we obtain the T-mappings

\[
\begin{align*}
    T(a_1) &= b_z^{-1}(d_s + d_z \psi)a_2 + b_z^{-1}d_z a_1, \\
    T(a_2) &= (e_z + f_z)a_1 + (f_z \psi + f_s)a_2, \\
    T(m) &= e_s + e_z \psi + (f_z \psi + f_s)m, \\
    T(\psi) &= b_z^{-1}(d_s + d_z \psi m - b_s).
\end{align*}
\]

For E-stability, the two required Jacobian matrices are

\[
\begin{pmatrix}
    b_z^{-1}d_z & b_z^{-1}(d_s + d_z \psi) \\
    e_z + f_z & f_z \psi + f_s
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
    I \otimes (f_z \psi + f_s) & I \otimes e_z + m' \otimes f_s \\
    I \otimes (b_z^{-1}d_s + b_z^{-1}d_z \psi) & m' \otimes b_z^{-1}d_z
\end{pmatrix}.
\]

References


